Special Properties of Private Equity in the Context of Portfolio Optimization

DISSERTATION
of the University of St. Gallen,
School of Management,
Economics, Law, Social Sciences
and International Affairs
to obtain the title of
Doctor of Philosophy in Management

submitted by

Mattias Thomas Fürstenberger

from

Germany

Approved on the application of

Prof. Dr. Heinz Müller

and

Prof. Dr. Christian Keuschnigg

Dissertation no. 3891

Medium GmbH, Lahr, 2011

The University of St. Gallen, School of Management, Economics, Law, Social Sciences and International Affairs hereby consents to the printing of the present dissertation, without hereby expressing any opinion on the views herein expressed.

St. Gallen, May 13, 2011

The President:

Prof. Dr. Thomas Bieger

Acknowledgments

> Wege entstehen dadurch, dass man sie geht.
> - Franz Kafka

Oder wie eine bekannte Volksweisheit sagt: Auf ausgetretenen Pfaden kommt man nur dort an, wo andere schon gewesen sind. Das Verfassen einer Dissertation erfordert das Beschreiten neuer Wege und die Überwindung zahlreicher Hindernisse und Sackgassen. Für die Begleitung während dieser Zeit bin ich einigen Personen zu tiefster Dankbarkeit verpflichtet, ohne deren Zutun diese Arbeit nicht möglich gewesen wäre.

In erster Linie möchte ich meinem Betreuer Herrn Prof. Dr. Heinz Müller von ganzem Herzen meinen Dank aussprechen. Er hatte immer ein offenes Ohr und stand mir stets mit seinem wertvollen Rat zur Seite. Seine Unterstützung war von unschätzbarem Wert. Ferner danken möchte ich meinem Koreferenten Herrn Prof. Dr. Christian Keuschnigg für die sehr angenehme Zusammenarbeit. Ein weiterer Dank geht an Frau Prof. Dr. Margrit Gauglhofer. Mit ihr fing bereits während des ersten Studienjahres alles an. Vielen Dank Margrit für Dein Vertrauen.

Ein ganz besonderer Dank gilt auch meinen Kollegen bei der Firma WaKa Holding AG. Vielen Dank für die ausserordentlich angenehme und inspirierende Zusammenarbeit. Eure Rücksicht und Euer Verständnis waren alles andere als selbstverständlich. Ganz besonders möchte ich Herrn Rolf Walser danken, der mir in diesen Jahren zum Mentor wurde. Vielen Dank Rolf für die guten Gespräche, Deine wertvolle Unterstützung und Dein Vertrauen. Im besonderen möchte ich auch Herrn Christoph Baldegger und Herrn Walter Kälin danken.

Mein aufrichtiger Dank gilt auch meinem Mentor Herrn Dr. Clemens Willée. Er hatte über die ganzen Jahre stets ein offenes Ohr für mich.

Vielen Dank Clemens für diese in keinster Weise selbstverständliche Unterstützung.

Besonders dankbar bin ich Thomas Ziegler für seine Kommentare zur Lesbarkeit des Manuskriptes. Vielen Dank, dass Du mir stets zur Seite standest. Ein weiterer besonderer Dank geht an Melanie Rudolf für ihre Kommentare zur englischen Sprache. Selbstverständlich sind alle verbleibenden Fehler ausschliesslich meine eigenen.

Danken möchte ich auch meinen Freunden, auf deren Verständnis ich immer zählen konnte. Vielen Dank für Eure Geduld.

Ein ganz besonderer Dank geht an meine Lebenspartnerin Deborah Schaub. Du warst immer für mich da und dafür bin ich Dir von ganzem Herzen dankbar.

Zum Schluss gilt meine tiefste Dankbarkeit meinen Eltern Karina und Karlheinz Fürstenberger für ihre bedingungslose Unterstützung und ihr unbeirrtes Vertrauen. Ohne Euch wäre alles nicht möglich gewesen und Eure Hilfe lässt sich nicht in Worte fassen.

St. Gallen, Mai 2011

Mattias Thomas Fürstenberger

Contents

Abstract

The following study develops a continuous time model evaluating the return loss from the delayed investment flow into Private Equity investments. Using several assumptions, an optimal rule to invest the committed but not invested capital can be derived analytically. The model context is extended to diversified portfolios consisting of Private Equity, stocks and risk-free bonds and an optimal investment rule is derived over time.

The fund flow into a Private Equity investment over time depends on the availability of investment opportunities. As a result, an investor cannot invest the entire commitment initially. The delayed investment flow results in a loss on the overall expected return of the Private Equity investment. If an investor cannot meet a capital call, he commits a default on commitment which is associated to a high penalty. The model derived throughout the analysis evaluates the opportunity cost from delayed investment and develops an optimal investment rule, taking the probability of a default on commitment into account. Extending the model setting to a diversified portfolio of Private Equity, public equity and risk-free bonds shows that the shortfall probability is close to zero for weights of Private Equity smaller than 50% for reasonable parameter values. As a result, the analytically derived optimal weights hold for those portfolios and the return loss can be reduced to a large extent.

In a second step, a continuous time model is derived and solved taking the specific characteristics of Private Equity funds into account. This results in optimization rules that are considerably different from standard cases. Most important is the fact that risky assets are massively overweighted especially concerning public equity if the investment delay is not taken into account. Also Private Equity adds additional risk to the portfolio consistent with the risk aversion of the investor. The results are very sensitive to the correlation structure. For higher levels of correlation Private Equity provides a good diversification instrument to public equity and in case of no correlation it is a perfect substitute for risk-free bonds.

Zusammenfassung

In der vorliegenden Studie wird ein Modell in kontinuierlicher Zeit entwickelt, welches den sukzessiven Investitionsfluss in Private Equity Fonds berücksichtigt und den daraus resultierenden Renditeverlust quantifiziert. Eine optimale Investitionsentscheidung zur Minimierung dieses Renditeverlustes wird analytisch hergeleitet. Zusätzlich wird das Model erweitert und eine optimale Allokationsregel für ein diversifiziertes Portfolio aus Private Equity, Aktien und risikolosen Anlagen entwickelt.

Die Investitionen in Private Equity Fonds erfolgen anhand der verfügbaren Investitionsmöglichkeiten des Fonds, d.h. das dem Fonds zugesprochene Kapital wird sukzessive abgerufen. Dies hat einen Verlust bezüglich der erwarteten Gesamtrendite des Private Equity Fonds zur Folge. Falls ein Anleger einer anberaumten Zahlung nicht Folge leisten kann, führt dies zu hohen Strafzahlungen. Das hier vorgestellte Model quantifiziert diesen Renditeverlust und leitet eine optimale Investitionsentscheidung für das noch nicht investierte Kapital her, welche diesen Verlust minimiert. Dabei wird die Ausfallwahrscheinlichkeit berücksichtigt. Bei einer Erweiterung des Modells auf Portfolios aus Private Equity, Aktien und risikolosen Anlagen resultiert eine Ausfallwahrscheinlichkeit grösser als Null nur bei Gewichten von Private Equity grösser als ca. 50% für realistische Parameterwerte. Daher ist die analytische Lösung auch für solche Portfolios optimal.

In einem zweiten Schritt wird ein Modell in kontinuierlicher Zeit entwickelt. Dabei werden die Eigenheiten von Private Equity Fonds explizit berücksichtigt. Die resultierenden optimalen Investitionsentscheidungen unterscheiden sich markant von herkömmlichen Ansätzen. Grundsätzlich werden riskante Anlagen übergewichtet, vor allem Aktien. Aber auch Private Equity wird konsequent übergewichtet, unabhängig von der Risikopräferenz des Investors. Die Ergebnisse sind sehr sensitiv gegenüber Veränderungen in der Korrelationsstruktur. Für höhere Korrelationswerte stellt Private Equity ein Diversifikationsinstrument für Aktien dar, im Falle fehlender Korrelation ein Substitut für risikolose Anlagen.

List of Figures

List of Tables

List of Abbreviations

Avg	Average
BM	Benchmark
BO	Buyout
CalPERS	California Public Employees' Retirement System
CAPM	Capital Asset Pricing Model
CARA	Constant Absolute Risk Aversion
CCAPM	Consumption Capital Asset Pricing Model
CRRA	Constant Relative Risk Aversion
DCC	Distributed to Committed Capital Ratio
det	Determinant
DPI	Distributed to Paid in Capital
e.g.	exempli gratia, for example
etc.	et cetera, and so on
excl	excluding
FOC	First Order Conditions
GDP	Growth Domestic Product
GMM	Generalized Method of Moments
GOP	Growth Optimum Portfolio
HJB	Hamilton Jacobi Bellman
ICM	Index Comparison Method
i.e.	id est, that is
IES	Intertemporal elasticity of substitution
incl	including
Inv	Investment
IPO	Initial Public Offering
IRR	Internal Rate of Return
mio	Million
NAV	Net Asset Value
NBM	No Benchmark
NG	No Investment Gap
No	Number
NSF	No Shortfall

oc	Opportunity Cost
p.a.	per annum
PICC	Paid in to Committed Capital Ratio
PME	Public Market Equivalent
RP	Representation Problem
RVPI	Residual Value to Paid in Capital Ratio
S&P	Standard & Poors
S.D.	Standard deviation
SF	Shortfall
SOP	Static Optimization Problem
SR	Sharpe Ratio
s.t.	subjet to
tot	total
TVE	Thomson Venture Economics
TVPI	Total Value to Paid in Capital Ratio
US	United States
USD	US Dollar
VC	Venture Capital
vs	versus

List of Notations

a	Liquid fraction of total wealth
c	Fixed annual payout
h	Fraction of risky asset meeting liquidity needs
k	Reduction of initial wealth
p	Opportunity cost from delayed investment
r	Risk-free rate
t	Point of time
t_0	Inception of Private Equity fund
t_1	Point of time for second investment round
$\boldsymbol{w}$	Vector of investment weights
x, s, b	Investment weights in mixed portfolio
$\bar{x}$, $\hat{x}$, $\tilde{x}$, y	Modified investment weights
z	Upper limit of normal distribution
A	Price of alternative assets
B	Price of risk-free asset
C	Price of Private Equity investment
$\bar{C}$	Price of not yet invested Private Equity commitment
F	Ratio of liquid assets to total wealth
N	Normal distribution
P_{SF}	Probability of a default on commitment
R^2	Share of total variance explained by a regression model
S	Price of public equity
T	Life-time of a Private Equity investment (investment horizon)
V	Covariance matrix
W	Effective wealth
$\bar{W}$	Theoretical wealth without investment delay
$\hat{W}$	Estimated wealth
$\tilde{W}$	Modified wealth
X	Liquid assets
Z	Brownian motion

F(.)	Optimization function
u(.)	Utility function
α	Jensen's Alpha, Penalty in case of a default
β	Systematic, asset specific risk
δ	Initial investment in Private Equity
δ_A, δ_S	Variance of the priors on risk-premia
ϕ	Expected value of the priors
γ	Coefficient of relative risk-aversion
μ	Expected return of risky assets
$\boldsymbol{\mu}$	Vector of risk-premia
π	Investment weight of risky asset of a Private Equity-only investor
π^*_M	Optimal investment of risky assets following Merton
θ	Vector of Sharpe Ratios
ρ	Correlation coefficient
σ	Standard deviation of risky assets
τ	Time period
ψ	Intertemporal elasticity of substitution
Δ	Covariance matrix of priors
Φ	Density of normal distribution

1 Introduction

Private Equity became one of the most interesting and most widely discussed asset classes in recent years. One mega deal follows the other and deal volumes jumped from one peak to the next increasing up to 40-50 Billion USD before the financial crisis. History has seen Private Equity investments since the 19th century when rich individuals invested their wealth in different kinds of projects mainly in the capital intensive areas of infrastructure and transportation. As the birth of the modern Private Equity industry, the foundation of the two first venture capital firms in 1946 is considered, one of them still existing today. Since the late 80s, the industry developed rapidly and large Private Equity funds emerged. This made pooling of capital possible and gave rise to larger deal volumes. On the other hand, modern Private Equity fund structures enabled investors to distribute their direct investments over several vehicles improving diversification and the risk-return structure of the portfolio. Today, there is an ongoing search for investments with low or even negative correlation to standard asset classes. During the last years, especially large US endowment funds were pacemaker of direct investment vehicles. With the availability of Private Equity investments to a broader group of investors it became also an issue in portfolio optimization.

Research in the area of Private Equity investments comes from many different disciplines and considers the field from as many viewpoints. In the area of portfolio optimization, Private Equity did not receive as much attention as could be expected taking into account the importance of Private Equity investments for larger portfolios and its omnipresence in the media. Especially in continuous time portfolio optimization there are so far no models looking at the special features of Private Equity investments. The investment class is included into the broad field of alternative investments together with hedge funds, quantitative funds, real estate etc. This study embeds Private Equity in an investor's continuous time portfolio optimization problem explicitly taking its special characteristics into account. There are recent studies that include alternative assets in the theoretical portfolio optimization framework but they have one thing in common: the amount of wealth invested in Private Equity is considered to be known. But Private Equity fund managers only invest the committed capital as they find suitable projects. This can delay the point in time when the full commitment is invested by a couple of years. The fact that committed capital is not equal to invested capital from the beginning is not included in the models, so far. Especially at the early stage of a fund, this difference can be substantial. As Private Equity is still a rather illiquid asset class and only open to a certain group of qualified investors, it is of special interest for an investor to optimally invest his capital, as committed capital cannot be split in parts and not be carried forward easily. This study quantifies the special nature of Private Equity investments and determines strategies on optimal investments applying a theoretical optimization framework on Private Equity investments.

1.1 Discussion of the Topic

The problem analyzed in this study consists of two parts: one is to invest the committed but not yet invested capital optimally and the other is to allocate Private Equity in the overall portfolio optimally.

The first comes from the special feature of Private Equity investments: the committed capital is not invested entirely upfront at the inception date of the fund but called by the fund manager during the investment phase which lasts normally up to 5-6 years and unknown ex ante. This delay can reduce the overall return on committed capital of the Private Equity investment to a large extent. As a result, the investor has to optimize the investment of that part of committed capital that cannot be invested in Private Equity from the beginning. Otherwise he gives away return opportunities. As he is forced to invest when a call on commitment occurs, the investor faces a tradeoff between investing optimally and a default on commitment. If the investor holds all the capital in cash, he gives away return opportunities. If he does not hold all the committed money in cash, he runs the risk of default on commitment as he might not have the required money for the next capital call. In this study, a model is developed taking this tradeoff into account and deriving an optimal investment path for the investment gap.

Another issue related to the successive investment flow into Private Equity investments is the optimal weight of Private Equity in the overall portfolio. Due to the tradeoff described above, it is unrealistic to assume that committed and invested capital are equal. As will be shown in this study, this leads to suboptimal portfolio weights. We assume that the Private Equity investment is split in two time periods. During the first, the investor is not fully invested in Private Equity and pays in the remaining part of the commitment at the beginning of the second period. The investor is then faced with the problem that he has to fix the portfolio weight of Private Equity already at the inception of the fund without knowing the exact investment weights ex ante. In the second period he is forced to invest the remaining committed capital increasing the weight of Private Equity in the portfolio massively without having the opportunity to reoptimize. This forces the investor into a suboptimal portfolio. In the following analysis a model in continuous time will be developed and solved that accounts for this problem and derives optimal portfolio weights analytically.

As a result, research on this topic is not only interesting from a theoretical point of view but also helps investors to get an overall and more comprehensive view of the asset class Private Equity. It gives advice how to allocate Private Equity within a diversified portfolio with the goal to maximize the respective expected utility of the investment based on quantitative considerations.

1.2 Structure

The structure of the study is as follows: After this short introduction, an overview of Private Equity as an asset class follows in chapter 2. The relevant features of the Private Equity industry will be discussed as well as investment rules and terms and conditions influencing risk, return and the optimization behavior of an investor. Furthermore, the investment flow into Private Equity funds will be analyzed and an overview on risk and return profiles of Private Equity funds will be given. A review on the special challenges measuring the performance of Private Equity investments before the exit concludes the chapter. In chapter 3, the relevant research on portfolio optimization including alternative assets will be discussed. It provides a short overview on the methods to derive the famous Merton solution and gives insights to continuous time portfolio optimization models that include alternative assets explicitly in the analysis. The model setting that will be used throughout the analysis will be introduced. We will solve a model analytically, that includes alternative assets but does not yet include the special features of Private Equity investments in order to derive a benchmark for the analysis in later chapters. In chapter 4, a model is developed that quantifies the opportunity cost of the delayed investment path into Private Equity investments. Under some simplifying assumptions an analytical solution to the optimal investment of committed but not yet invested capital is derived for an investor who invests his entire wealth in Private Equity. The chapter includes a discussion of the probability of a default on commitment and concludes with a simulation of opportunity cost without the assumptions made before. In chapter 5, the model is extended to cases where Private Equity is part of a broad portfolio of Private Equity, stocks and risk-free bonds. The optimal investment paths for several cases are simulated taking the delayed investment path and the probability of a default on commitment explicitly into account. Common liquidity constraints that institutional investors often face are discussed within the context of the model. Furthermore, a continuous time portfolio optimization model is derived and solved analytically considering the investment delay into Private Equity funds. The results are compared to the benchmark case derived in chapter 3. Chapter 6 concludes the study and provides an outlook for further research. Information on data, institutional issues and mathematical derivations is given in the Appendix.

2 Review of Private Equity

2.1 Overview of the Private Equity Industry

The Private Equity industry has grown enormously over the past 20 years and was one of the fastest growing asset classes comprising lots of different types with venture capital, buyout capital and mezzanine capital among the most important. The by far most important data source, which is used in numerous studies mentioned later and also in this study, is the *Thomson Venture Economics (TVE)* dataset. It includes data of over 15'000 funds and detailed information on the legal structure of each fund and its several investments. The cash flow data is net of fees as it is from the investor's perspective. But the data has some shortcomings: it is available only in aggregate form, it is self-reported and so prone to be biased and it also includes unrealized investments what may give rise to accounting biases. Comments on this issue will follow in later chapters. The following two graphs (2.1 & 2.2) show the evolution of the number of Private Equity funds and the investment volume since the beginning of Private Equity in the late 60s to the end of 2008.

As we can see from both graphs, Private Equity as an asset class started in the mid 80s to evolve and we observe the big growth rates only from the early 90s onwards where we have a sudden and massive upsurge in both the number of funds and the capital raised by those funds. The two graphs show both the overall data and the distribution between venture capital (VC) and buyout (BO) funds, the by far most important types of funds (about 80 % of capital invested). If the breakdown of the figures is compared, one will notice that the number of funds is mainly driven by the upward rocketing number of VC funds. Nevertheless, BO funds

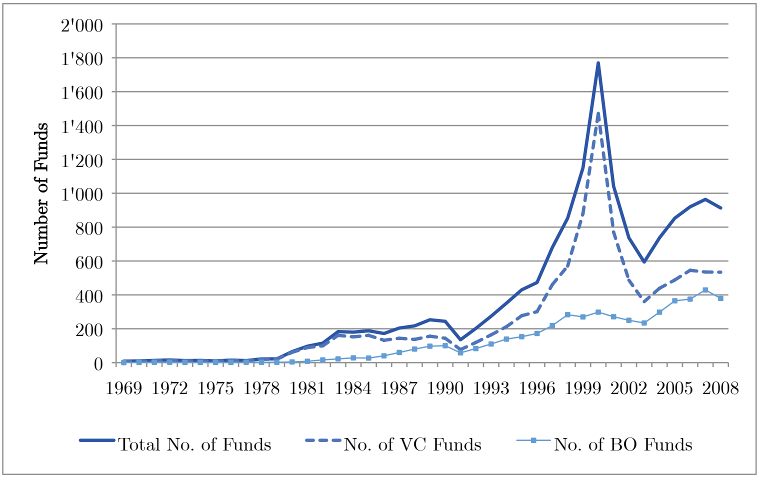


Figure 2.1: Number of funds (Source: TVE)

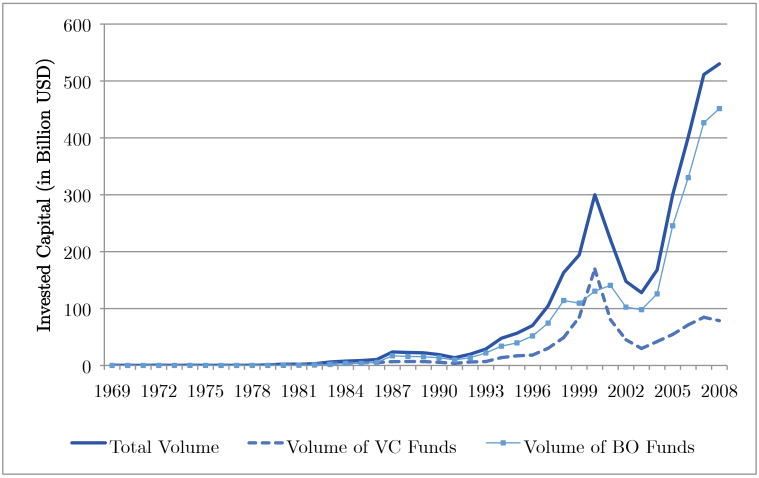


Figure 2.2: Investment volume (Source: TVE)

almost doubled within the last ten years. For the investment volume the relation is vice versa. Buyout funds dominated the overall capital in Private Equity funds to a large extend especially after 2003 absorbing more than 85% of total Private Equity capital up to 2008.

Figure 2.3 visualizes the spread between VC and BO funds by showing the average fund size. The average fund size for VC funds was quite stable across the last decades. BO funds started to grow bigger especially in 2005 when money inflow experienced large growth rates.

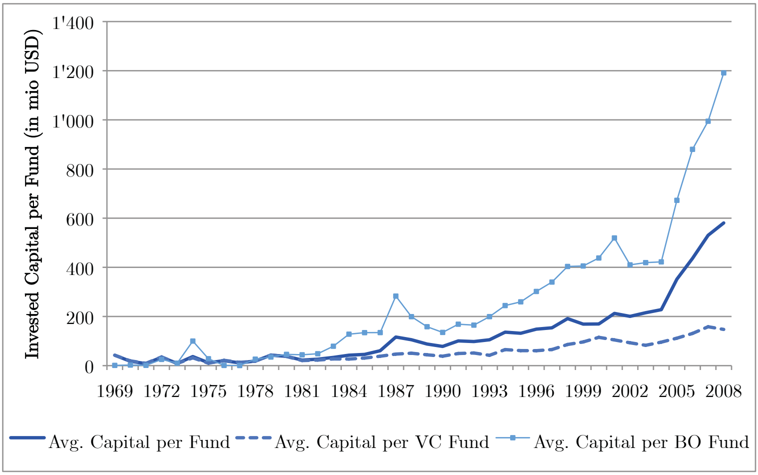


Figure 2.3: Average capital per fund (Source: TVE)

The effects of the dotcom-bubble in 2000 can be seen clearly from both figures 2.1 and 2.2 especially for VC funds as most of the dotcoms were small start-ups funded with venture capital. The impacts of the financial crisis are especially visible from the fundraising figures. According to Prequin (2010a), capital raised in 2010 was 225 Billion USD compared to 530 Billion in 2005. Especially in 2009 capital inflow was only half the level of 2008. The difficulty to raise money in the aftermath of the financial crisis can also be seen from the average time it takes for a fund to have its final close, i.e. to raise the targeted amount of money[1]. The funds having closed in 2010 needed on average 20.4 months to collect

[1] Final close means closed for further commitments prior to starting the investment phase.

the capital up from 14.9 months in 2008 and 10.6 months in 2005. The problems to raise money during and after the financial crisis mainly come from the illiquid nature of Private Equity funds. The outlook for 2011 is somewhat better. According to Prequin (2010a), 33% of the investors want to increase their share of Private Equity in the portfolio and even 37% want to increase it over longer terms. But in 2010 the vast majority of investors is at or above the target level of Private Equity. On the other hand, since the financial crisis more than 60% of the investors expect an outperformance of Private Equity ranging above 4% compared to the public market.

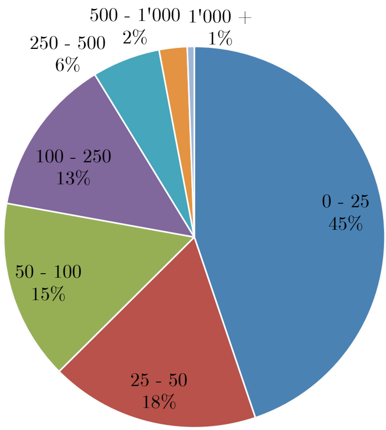


Figure 2.4: Venture capital fund size (in mio USD, Source: TVE)

There are also differences between VC and BO funds with respect to fund size. While funds between 100 mio USD and 1 Billion USD make up about 65% of the capitalization for VC funds with quite equal distribution, BO funds are larger than 1 Billion USD for more than 63% of the total capitalization. Figures 2.4 and 2.5 show the distribution of Private Equity fund sizes by March 2009 as a percentage of total funds. The results differ compared to the capitalization-based figures mentioned above. While BO funds are quite equally distributed over fund size, almost 50%

of the VC funds are smaller than 25 mio USD. Funds larger than 250 mio USD make up less than 10% of total funds. As VC funds mainly invest in smaller firms or even start-ups which have less need in absolute amounts of capital per investment, fund sizes are as expected. It becomes also clear from the *TVE* dataset that the fund size increases on average with the development stage of a funds targeted investment objects. Other fund types with disproportionately large fund sizes are secondary funds and turnaround/distressed-debt funds.

The structure of investors is very similar for both fund types being mainly institutionals and high net worth individuals. More than 70% of both types of funds are organized as "Independent Private Partnerships". The only difference is that for BO funds there are 3% of funds doing secondary purchases. For VC funds corporates have also some importance as investors. Figures 2.6 and 2.7 show the different types of investors in the two respective fund classes.

The number of a fund's transactions decreased massively after the financial crisis being only 16.5% in 2009 compared to the level in 2006.

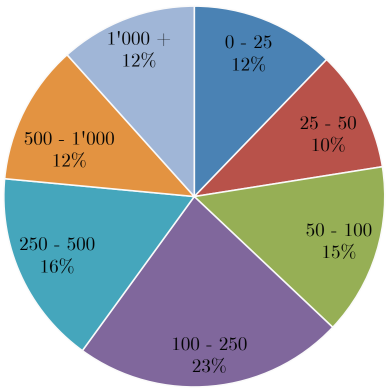
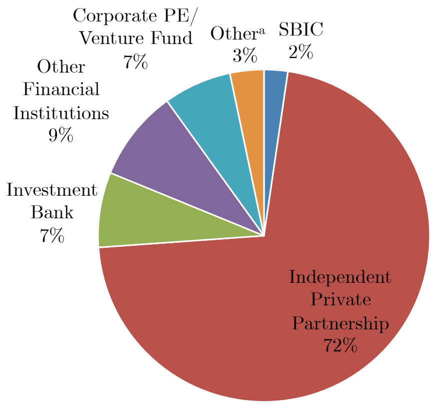
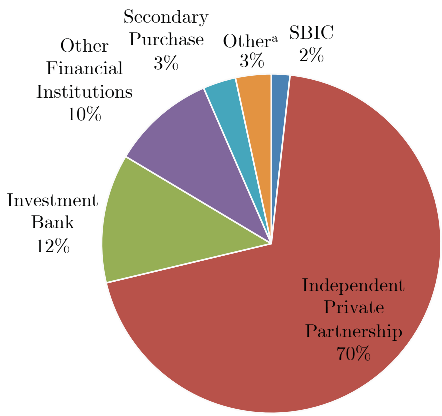


Figure 2.5: Buyout fund size (in mio USD, Source: TVE)



[a] Other includes: Investment Advisory Affiliates, Government, Development Programs, Angels, Individuals, Evergreen, Endowment Funds and Foundations

Figure 2.6: Venture capital fund investors (Source: TVE)



[a] Other includes: Investment Advisory Affiliates, Government, Development Programs, Fund of Funds, Angels, Individuals and Evergreen

Figure 2.7: Buyout fund investors (Source: TVE)

In 2010, the level even fell to 5.71% indicating an extremely inactive market. Average value per deal reduced by about one-half and the overall value of all completed deals was in 2010 only 2.5% the level of 2006. But as these reductions do not come from the attractiveness of Private Equity as an asset class itself, but more from the overall need of liquidity in the aftermath of the financial crisis a recovery of the investment flow into Private Equity is likely to happen. The results from the surveys conducted by Prequin also supports this conclusion[2].

2.2 Private Equity as an Asset Class

2.2.1 Outline of Typical Private Equity Funds

This study does not consider a special sub-category of Private Equity but focuses on the typical structure of a Private Equity fund. We constrain ourselves to venture capital and buyout funds as the most common types. The tradeoff for all Private Equity investments is liquidity versus taking advantage of imperfect information and niche investment opportunities. As Private Equity investors acquire specific and not easily observable knowledge, the investments are normally not traded frequently due to pricing difficulties and their legal structure and are therefore very illiquid.

Usually, a Private Equity fund consists of general and limited partners. The general partner, normally a specialized Private Equity firm, is the manager of the fund and the limited partners are the investors. When a new fund is introduced, the investors commit a certain amount of money up to the maximum amount the general partner wants to raise. After the subscription period, the general partner draws the money he needs for the first investments. Further capital can be drawn at any time during a predefined investment phase up to the maximum commitment of each investor. The speed mainly depends on the ability of the general partner to find appropriate investment opportunities but is also reduced by the exit

[2] For a comprehensive overview on the Private Equity industry in general the reader is referred to Gilligan and Wright (2010).

rates of existing investments. Sometimes the capital paid back during the investment stage of the fund is used to fund new investments. Therefore, there are a lot of random sources influencing future capital calls and, in line with this, invested capital.

The lifetime of a fund is normally fixed to ten years with the option to extend it up to four years. Fees mainly consist of a fixed management fee and a fraction of profits, the carried interest. In order to maximize carried interest, fund managers have an economic incentive to invest all committed capital but as they also participate in final exit revenues, they have an incentive to screen carefully and invest only in profitable projects. Drawing more than has been committed is often prohibited. Typically, the fund can make new investments only during the first five years but follow-on investments can be made for the whole period. The fee structure can be very different across funds and is sometimes difficult to compute due to its proportional character[3]. An interesting analysis on the fee structure of a Private Equity fund is made in Connor (2005a) where the economic value of several terms and conditions is evaluated by estimating final portfolio values excluding several terms and holding all others constant[4]. He found some terms to be of only minor importance for a fund's return expectations like preferred return, clawbacks in case there is a return provision on capital and fees, catch-up provisions or the calculation base of carried interests. The level of carried interest itself has by far the most important impact on returns. As a result, investors should try to avoid these cost but insist more on terms reducing their opportunity cost of late investment which can decrease their return by considerable amounts as will be shown in later chapters.

Normally, Private Equity investments require a large minimum payment (commitment) and are open only to qualified investors. Together with the facts that transferring shares of a Private Equity fund is always subject to the approval of the general manager and the lack of market prices this

[3] For a detailed analysis of the fee structure and the cash flows to the general and limited partners the reader is referred to Connor (2005a) and Metrick and Yasuda (2008).

[4] For an overview on the standard terms and conditions the reader is referred to Appendix B.2.

is the main reason why the illiquidity in the Private Equity market exists. The difficulty to agree on fair values will be discussed later. The procedure in case that a default on commitment occurs is typically outlined in the fund's organizational documents. There is typically a transition period where capital commitments can still be installed after the payment was due. During this period, the limited partner has to pay penalty interests and an administration fee for late payment. A fixed period after the notice of default the investor failing to pay will be considered a defaulting investor. In case of a default, an investor will normally not be entitled to make any further contributions to the fund. He will remain liable for the full unpaid portion of the commitment unless he is able and allowed to transfer either this liability or the whole stake in the fund. The defaulting investor will not be entitled to receive any distributions associated with investments made by the fund after the default date. In case of a default, the general partner is allowed to solicit offers for the defaulting investor's stake in the fund. Often it is limited in a first round to the other limited partners. In most cases, the price cannot be below a certain fixed discount of the fair market value (often around 25%). As the market value is determined by the general partner, this lower limit can be rather theoretical and subjective. If no buyer can be found after a certain period, the stake of the defaulting investor is reduced by the fixed discount and the rest is allocated to the remaining limited partners. Distributions concerning the investments made before the default date and still attributable to the defaulting investor will be collected in a separate account and only paid to the defaulting investor after interests and fees are subtracted at the end of the fund's lifetime[5]. Regarding the rather huge loss in the case of a default and the illiquid character of Private Equity investments, the necessity of pricing default risks when optimizing a portfolio including Private Equity becomes evident. As there is still a lack of theoretical considerations on this aspect of Private Equity investments, this study will

[5]These informations come from the terms and conditions of several Private Equity managers that do not want to be named. An overview on general terms and conditions is given in Appendix B.2. An overview on the characteristics of Private Equity funds is also given in Gilligan and Wright (2010, p. 29ff).

close the gap and help investors to value risk and return of their overall portfolio correctly.

2.2.2 Description of Private Equity Returns

To get an understanding on the performance of Private Equity funds, the aggregated data from *TVE* is analyzed first. Between the first measuring in 1969 up to June 2010, VC funds returned on average 7.82% p.a. and BO funds 9.42% (measured as IRR net of fees). With capital weighting these numbers shrink to 5.37% and 3.68%. Of course, IRR figures between 2005 and 2010 are typically negative as those funds are still in the investment phase. The numbers are nevertheless included here to incorporate the typical J-curve character of Private Equity returns, which also influences an investor's annual portfolio return[6]. From these figures, we can infer several findings. First, it looks somewhat curious that VC returns are not much lower compared to BO returns and even larger capital weighted despite of the positive leverage effect that drives BO funds' returns. If we look at the median returns, the picture turns to what could be expected. VC median returns are 1.45% and BO median returns 6.74%. This is due to the tremendous spread we observe especially for VC fund returns with incredibly high gains and total losses which is not that distinct for BO funds. The second implication is the spread between average returns and capital weighted returns which shows that smaller funds performed better on average. Ljungqvist and Richardson (2003) also found in their study fund size to be a negative and statistically significant driver but on a very low relevance level R^2. The picture for US and European VC and BO funds looks similar but with lower return figures and a very pronounced return gap between small and large European BO funds. The return differences across different types of VC funds is not very disperse, but funds investing in early stage companies tend to perform best on average. For BO funds there are practically no differences across different types besides mega-buyout funds which performed extremely poor on average. Accord-

[6] J-curve effect means that returns over a fund's lifetime follow the pattern of a J-curve: going down in the first years when fees, especially management fees that are paid on committed capital, dominate distributions. It might take several years for a fund to show positive returns, normally when the majority of distributions are realized.

ing to Prequin (2010b), more than 50% of the investors consider small and medium buyout funds to offer the most promising opportunities in 2011.

Comparing the figures above to IRRs as of December 2008 gives some insights on the effects of the financial crisis on Private Equity returns. The data shows that large funds performed better during the crisis as average returns are lower in 2008 for both VC and BO funds. With capital weighting this effect is much more pronounced especially for BO funds indicating that large funds performed better than smaller funds. For BO funds capital weighted average returns before the crisis were even negative (-4.48%).

The second dataset that is analyzed is the alternative investments part of the CalPERS portfolio[7]. The portfolio includes all kinds of Private Equity funds with buyout capital being the largest group (68%) followed by venture capital (9%). We find an average IRR since the foundation of the portfolio in 1990 to 2010 of 7.71%. If we consider only the performance between 1990 and 2005 which excludes the funds still in investment phase, we observe an average IRR of 14.15% and 15.24% capital weighted. These figures show the tremendous impact of including funds in the early stage into the return measures. For the CalPERS portfolio the fact that smaller funds tend to have a higher return does not hold. But there is another interesting finding that can be inferred from the CalPERS dataset. The return figures reported above are as of June 30, 2010. Comparing these figures to the returns retrieved as of September 30, 2008 gives some insights on the performance of the portfolio in the financial crisis as the figures reported have a two-quarter lag. Returns of mature funds then were 18.05% and 17.01% capital weighted. These figures are in line with the findings above about fund size. Overall, the portfolio lost some return during the financial crisis and the fact that capital weighted returns did change less gives rise to the conclusion that large funds tended to struggle

[7]The California Public Employees' Retirement System (CalPERS) comprises 632 funds with 52.98 Billion USD of committed capital. CalPERS is legally obliged to publish all information of their Private Equity investment funds which is available on its website at www.calpers.gov.ca. A detailed description of the portfolio is provided in Appendix A.

less during the financial crisis. An explanation might be that large funds can invest in larger companies that were more stable during the crisis. This is also supported by the fact that the CalPERS portfolio consists mainly of BO funds investing in more mature projects being less prone to struggle.

2.2.3 Describing and Modeling Private Equity Commitments

The cash flows in and out of a Private Equity investment are very uncertain. In this part, the focus will be on the rate at which the money is invested. It is very important to get a grasp on the timing of commitment calls because of the time value of money effect on the return of the investment and second because of the effect on investment/default risk.

The following figure 2.8 shows the typical cumulative draw down pattern of a Private Equity investment depending on its lifetime.

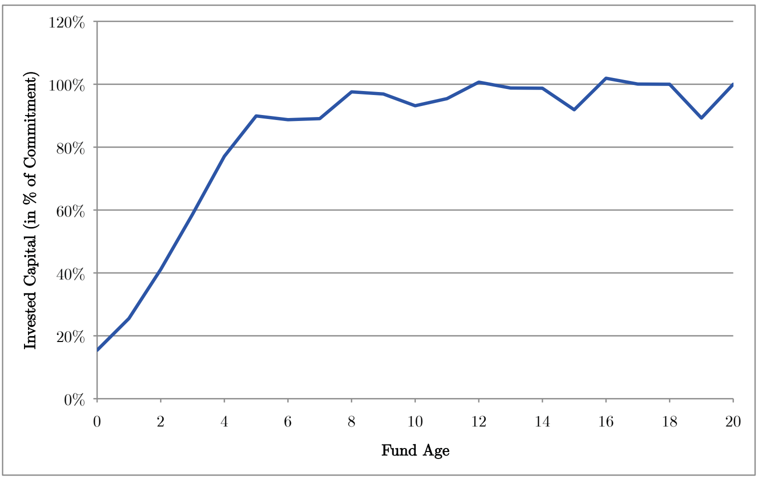


Figure 2.8: Average investment path of Private Equity funds (Source: CalPERS)

It takes about six years that more than 90% of committed capital will be drawn which is in line with the average investment period of normally 5-6 years. The majority of the funds does not invest anymore after year 6. 86% of the funds older then ten years were invested more than 90% but only 45% were fully invested. On the other hand, 4.35% of the 2009 funds were already fully invested at the end of the vintage year. Variance is only larger than 2% for still investing funds (vintage years: 2005-2010) and falls below 1% afterwards. For funds in the investment phase the variance can be substantial with a maximum of 7.14%. It is obvious that the fund age is a key determinant of expected capital calls. In most of the years, CalPERS was invested more than originally committed for several mature funds up to an overcommitment of 2.83%. Total overcommitment amounted to 85.31 mio USD being 0.19% of total assets in Private Equity. Overall, the investment ratio (invested/committed capital) lies around 95% for mature funds (older than 5 years). As a result, using a rolling forward investment strategy CalPERS is able to reach a level of commitment close to its target. But there is significant variance and there is always overcommitment in several funds further enforcing uncertainty. Therefore, despite using rolling investments CalPERS is left with a sum of on average 2.25 Billion USD which is committed to Private Equity Funds but never paid in. These funds have to be invested optimally which is the goal of the model derived later.

In Figure 2.9, the average investment degree across all funds is displayed over the last 25 years for US venture and buyout funds. Since the Private Equity industry became more established in the beginning of the 90s, the investment ratio was between 70-80% for VC and BO funds. The paths for both types moved very close since then but only reached a degree of 81% for VC funds and 75% for BO funds until 2008. Of course, these figures are downward biased, as recently founded funds are also included to build the average. But as a Private Equity investor normally holds funds across almost all vintage years, as we can see from CalPERS, the investment ratios shown in figure 2.9 provide a good approximation for the path of the investment degree over time. The studies cited below reveal very similar results[8].

As a result, these numbers show that the gap between committed and invested capital is a big issue for portfolio optimization considerations. If we consider the portfolio of CalPERS with the given weights of both fund types and a total commitment of 52.98 Billion USD, the investment gap in 2008 would amount to 12.95 Billion USD or 5.94% of totally managed assets which is a sum definitely too large to be not invested optimally.

There are studies that analyze the investment path of Private Equity investments and the underlying drivers in order to derive investment strategies that bring the invested capital closer to a predefined level and turn it from a stochastic variable into a (more) deterministic one. Given the highly uncertain nature of cash in- and outflows this is a very demanding task and can surely not rely on simple rules of thumb. The common view that an overcommitment strategy of 50% compared to the desired level can be the solution does not take the complex behavior of cash flows into account[9]. It is obvious that such a strategy can only be at the cost of increased shortfall risk which can lead to severe disadvantages in case of a default on commitment. If the rise in shortfall risk was priced in, this strategy would become very costly. Nevertheless, it is a common strategy that investors follow. There is a straightforward reason for this approach. As many investors see Private Equity only as one (minor) asset class within a large and well-diversified portfolio (CalPERS: 14%), there is practically no shortfall (default on commitment) risk when the whole portfolio is taken into account and therefore the costs almost equal zero.

This view might be correct from the perspective of the costs arising from larger shortfall risks, but from another perspective it is rather questionable: optimal investment behavior or foregone return opportunities. If a liquidity shortfall occurred, it might be suboptimal to withdraw money from another investment as the optimal investment path has to be left. As a result, if we take a comprehensive view on the overall portfolio, a

[8] For a more detailed analysis on the statistics of capital calls the reader is referred to Zwart, Frieser and Dijk (2007), Connor (2005c) and Frei and Studer (2003).

[9] There are also rules of thumb in the academic literature. View i.e. Cardie, Cattanach and Kelley (2000).

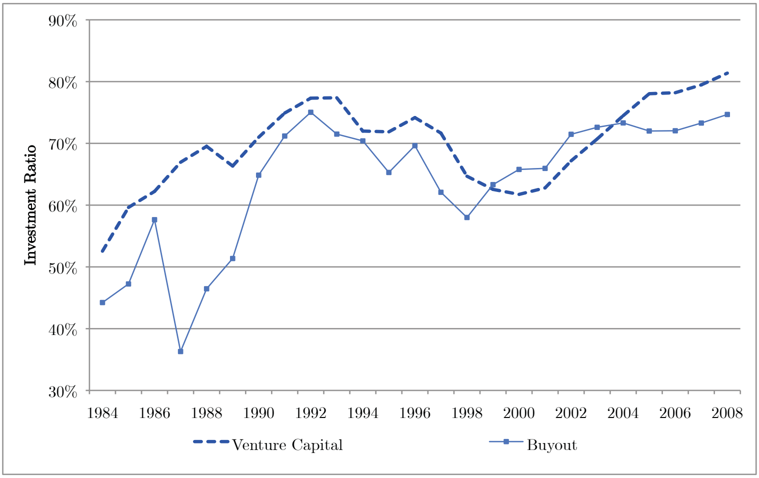


Figure 2.9: Average degree of investment (Source: TVE)

rigid overcommitment strategy can never come at any cost. The following chapters analyze these different effects, compare the arising costs and develop an optimal investment strategy. The central problem will always be the treatment of the investment gap, because it can never be possible to close it completely as the stochastic nature of invested capital can never be neutralized completely. So, minimizing the investment gap and higher shortfall risk are the flip side of the coin and the resulting cost have to be priced in and compared.

The most recent study on optimizing the investment path (minimizing the investment gap) taking into account shortfall risks is from Zwart et al. (2007). They develop a heuristic recommitment strategy taking into account the illiquid nature of Private Equity investments. Their strategy makes new commitments every quarter and is dynamic in the sense that it takes the current portfolio into account: paid out capital is recommitted immediately, capital not called after a certain period is also recommitted but multiplied with the reciprocal of the current investment degree to reduce the risk of overinvestment and liquidity shortfall. Testing this strategy for different fund types and across different regions, Zwart et al. (2007,

p. 14) manage to keep the invested capital close to its target and holding shortfall risks in reasonable bounds. The investment degree is 85% on average with a low standard deviation of 9% compared to 75% and 9.2% for the reference portfolio. This strategy also leads to overcommitments at several times. But due to its dynamic nature, shortfall risks can be kept at an average of 8.8% (median: 8.4%). The maximum overcommitment was 19%. Adding a fixed ratio of overcommitment to this strategy can push the investment degree up to 98% but only at the expense of an average shortfall risk of 41%, which makes the described tradeoff very clear. It can be concluded that the investment degree can be pushed further using appropriate strategies but the investment gap amounts still to 15% of committed capital. Another problem of this strategy is the need to find appropriate Private Equity investments at any time to make the reinvestment.

Another commitment strategy is provided by Connor, Nevins and McIntire (2004). They calculate a formula for a target commitment level based on the desired investment level including the expected rate of return on private and public investments and the commitment and exit rates. As the commitment rate is measured relative to its market value, they try to link it to past performance. As a result, an initial overcommitment of 70% was found together with an ongoing annual commitment rate of 2.8% for the long-run (p. 36). The obvious shortcoming of this approach is the assumption of fixed rates of return and capital flows into and out of Private Equity funds which is completely apart from the characteristics of Private Equity investments. The liquidity shortfall is ignored. For this reason an overinvestment of about 0.5% is observed which lasts for about 5 years. This number might seem quite low but if we apply this result to the CalPERS portfolio, the overinvestment amounts to 264.9 mio USD which has to be financed somehow over several years.

A different approach to optimize the commitment level is the direct estimation of cash flows and deriving an investment path from it. Frei and Studer (2003) describe two possibilities: the first is the estimation of cash flows for the underlying companies and aggregating them to the fund level and the second is the cash flow estimation at the fund level directly. Independent of the method used to conduct these estimations it is clear that both approaches suffer from substantial modeling risk.

An interesting model was developed by Dean and Seth (2002) with a different objective compared to the models described above. They use assumptions of several Private Equity fund characteristics and estimate capital calls, distributions and net asset values to calculate an IRR. They continuously update these estimates by including most recent data. Their goal is to optimally allocate commitments to the funds in their portfolio. So, the focus here does not lie directly on the optimal overall commitment level but on the allocation of the commitments across funds given a certain commitment level. Of course, it is clear that the random character of cash flows can also not be changed by this approach.

Figure 2.10 shows the actual and targeted weights of Private Equity in the portfolio of the Yale University Endowment Fund[10]. The Yale University Endowment Fund, among some other large US endowment funds, was one of the earliest investors in Private Equity and real assets. The graph shows clearly that despite professional commitment strategies and rolling investments targeted and invested capital differ on average by 2.34% and the overall investment ratio is 88.3%. If this amount is weighted by total assets under management, the right hand side of figure 2.10 shows the absolute amount that is committed but not invested averaging 300 mio USD per year with a maximum value of 728 mio USD. This amount is not invested optimally when the delayed investment flow into Private Equity is ignored. The graph also shows that when demand for Private Equity investments is rather high, as for example during the dotcom-bubble or during the sudden economic upswing after the financial crisis in 2009, it is more difficult for funds to invest and therefore actual versus targeted investment in the Yale portfolio differs more. During the financial crisis, investing was more easy but on the other hand market activity tended to be zero which explains the increase in 2008. These findings support the

[10] All data on the investment strategy of the Yale University Endowment Fund is available in the "Yale Endowment Updates" available from http://www.yale.edu/investments/.

data analyzed earlier and fit the explanations found in several empirical studies cited below.

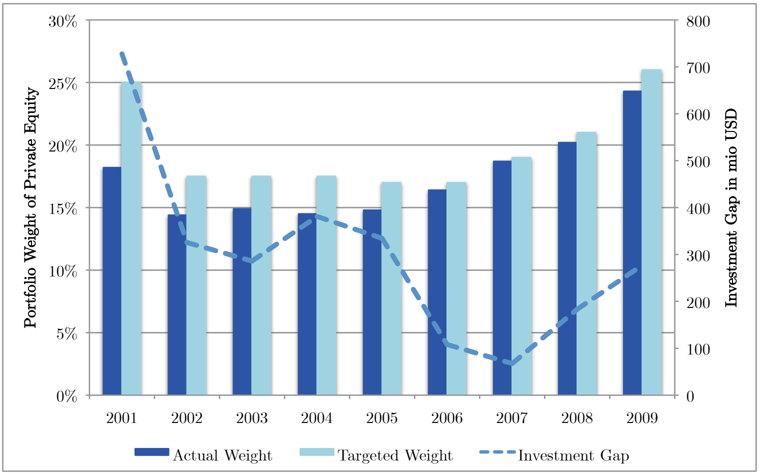


Figure 2.10: Private Equity weights in Yale portfolio (Source: Yale (2009) & (2005))

Ljungqvist and Richardson (2003) identify some factors influencing the speed of capital calls in their study. Venture funds take longer to invest than buyout funds but with only marginal significance. Funds raised between 1981 and 1993 invest much more rapidly. A reason might be the fact that Private Equity was a rather young asset class at that time and competition for "good" deals much lower. Also the actual fee structure and the resulting incentives was not developed yet. The availability of investment opportunities is also strongly significant. This view is also supported by the positive influence of the control variable for competition and the behavior of the investment degree displayed in figure 2.9 also affirms the argument. During the dotcom-bubble in 1999 to 2001, especially the number of VC funds and their capitalization increased tremendously. Therefore, competition was extremely high and the availability of good investment opportunities relative to the amount of inflowing capital rather low. Public market developments do not appear to influence the speed of capital calls which goes in line with the common understanding of Private Equity as being only moderately correlated with public markets. This finding is also supported by Connor (2005c, p. 4). Interesting is the fact that funds invest more slowly as debt becomes more expensive. But this finding might be spurred by the fact that the dataset is strongly biased towards buyout funds and therefore the generality of this point is rather questionable.

2.2.4 Return Structure of Private Equity Investments

Since the fundamental work of Jensen (1968) in estimating absolute returns of risky assets, a lot of effort has been made to improve the predictability of risks and returns of risky assets within different frameworks. But there is still a big ongoing debate on the performance of Private Equity funds. The comparability of the results is somewhat complicated because of the different data sources and return measures used. This is also widely considered as the main reason why results of different performance analyses are very inconsistent.

The first ones studying the returns of public and Private Equity investments were Moskowitz and Vissing-Jørgensen (2002). They used a large sample of individual Private Equity investments from several sources. They found Private Equity returns to be not higher than public equity returns. The main problem from their analysis results from the inclusion of self-employment and entrepreneurial investments to the dataset. The difference to Private Equity as a pure financial investment is the fact that entrepreneurial investments and self-employment decisions are not based completely on financial reasoning but also depend on social factors and intrinsic motivation. Therefore, the results cannot be compared to Private Equity funds as financial investments are normally part of a diversified portfolio. The fundamental study on Private Equity as a financial investment was conducted by Kaplan and Schoar (2005). They screened a portfolio of 746 Private Equity funds and found out that their portfolio on average yielded comparable returns net of fees to the *S&P 500 Index* with large variation in time. They also found large funds to perform somewhat better and beating the index on average (p. 33). They used the *TVE* dataset but used single fund data from different public sources[11] as well. Having in mind the common picture of the Private Equity industry yielding spectacular returns, which is often drawn by the world press, this seems to be surprisingly low[12]. On the other hand, Kaplan and Schoar (2005) also report a large heterogeneity of performance across funds which fits to the high-risk high-return characteristics of Private Equity investments. Gottschalg and Phalippou (2006, p. 17ff) even regard this performance as too high, arguing that the selection of funds in the *TVE* dataset is upward-biased. The main problem in measuring performance might lie in the nature of a fund's accounting standards. The value of existing investments is still in a fund's books although there is no more sign of activity and the regular lifetime of ten years is also exceeded (p. 12). They also criticize the weighting scheme which is oriented on committed and not on invested capital biasing towards funds having been successful in raising money but not in investing it. Their solution consists of including funds' numbers of successful exits as a proxy of performance and therefore only use funds older than ten years. Correcting for the shortcomings mentioned above, they indicate an underperformance of 12% relative to the performance of the *S&P 500* which holds for both venture capital and buyout funds.

For gross performance (before subtracting fees) the picture is mixed. While Cochrane (2005) gets an average return of 59% gross of fees for VC, Hwang, Quigley and Woodward (2005) only find moderate outperformance compared to the *S&P 500* and the *NASDAQ* gross of fees. The underlying datasets end in 2000 when we observed an extreme peak of the Private Equity industry in every respect. But in spite of the almost same datasets, the results are not even similar. Hwang et al. (2005) also conduct the same analysis for the dataset ending in 2003 and then did not even find outperformance gross of fees anymore. These results are somewhat difficult to interpret as funding data from individual deals are

[11] One of them was also the CalPERS dataset which was analyzed above.

[12] For a selection of related press articles the reader is referred to Appendix A in Gottschalg and Phalippou (2006).

used which are very difficult to handle because of their extremely random nature[13] and returns are not estimated from a sample of liquidated funds directly. One of the very few studies on buyout funds' performance was conducted by Swensen (2000, p. 230ff). He focused on risk-adjusted returns and found out that BO funds did massively underperform the *S&P 500* if the same Debt/Equity ratio is applied to the index. All of the other studies mentioned above found returns gross of fees to be substantially higher than for standard market indices. Metrick and Yasuda (2008) support this as they found out that the fixed part of the fees makes up a considerable share in a general partner's revenue. 64% for both venture capital and buyout funds. The overall fees are generally about 30% higher for venture capital funds. But buyout fund managers can profit more from their prior experience and increase the size of the funds more quickly. This is possible because buyout funds focus more on big and rather mature companies and so their business is more scalable.

Ljungqvist and Richardson (2003) also discovered some interesting features about Private Equity performance. They used cash flow data from a limited partner and had access to all single fund data. All investments made by the limited partner are included and so the survivorship bias can be overcome. This bias arises because publicly available return data only exist for finally closed funds which in general are the more successful ones as others are closed with a loss or written off. On the other hand, the dataset is affected by the investor's investment policy which in this case is highly focused on buyout funds. The main source of performance data on buyout funds is their track record which is mainly reported in fundraising brochures of the general manager (Phalippou, 2007, p. 4). Another advantage of the Ljungqvist-Richardson dataset is the fact that they have access to the cash flow data of all investments and also their respective valuation which improves the reliability of the IRR-figures used compared to the aggregate IRR reported in *TVE*. The average IRR for the mature funds in the dataset including fees was found to be 20.46% on average and 17.67% capital weighted with a standard deviation of 22.42%. So, larger

[13]The goal of the paper by Hwang et al. (2005) is different from the development of a simple return estimation. They construct a benchmarking index for VC returns.

funds performed somewhat worse. The large difference on performance can be mainly credited to the characteristics of the dataset which relies completely on historical cash flow data. Using the *S&P 500 Index* as a benchmark, Ljungqvist and Richardson (2003, p. 18) also found significant excess performance ranging from 5.93% to 8.06% depending on the assumptions on investment and de-investment speed. Compared to the *NASDAQ Composite Index*, the range reduces to 2.62-6.28%. Again, a substantial divergence of returns is observed. An interesting fact is that average IRRs only turn positive in the eighth year of investment and excess returns even later taking into account capital cost.

The survivorship problem is also actively controlled for in a study by Cochrane (2005) where he established a maximum likelihood estimation for gross returns. He identifies and measures the underlying return distribution, exit dates and increasing probabilities of getting a return as the asset value increases. Very interesting is the finding that the returns of venture capital investments are approximately log-normally distributed with a small mean and a high standard deviation. Mean-log-returns are found to be only slightly above the respective *S&P 500* mean-log-returns with a substantially higher standard deviation especially for early stage deals. But the standard deviation is comparable to small *NASDAQ* stocks. These findings show clearly that high average returns of venture capital investments mainly come from a high standard deviation not from high mean returns. As Cochrane examines individual venture capital projects, he estimates gross returns. He used financing data from *TVE* amended by public offering or M&A data from different sources. The correction results in a massive downsizing of average returns and their standard deviations. A major shortcoming of this analysis is of course the dataset which ends in 2000, the absolute peak in Private Equity in every respect which might cause some upward bias for the estimations.

If we summarize these findings we do not really see Private Equity funds outperforming public equity on average to a large extent. Phalippou (2007, p. 4) puts it very clear: "Even on a sample that is clearly biased towards winners, if the average performance is properly aggregated and the sample sufficiently large, then average performance is low!" The conclusion is obvious: We do find outperformance but only for several funds and experienced fund managers but not on average. Therefore only skilled investors able to detect the "good" funds are able to yield above average returns. This problem is also closely related to the famous agency theory as investors are unsure on the quality and the behavior of the fund manager.

If we think of possible performance drivers for Private Equity investments we end up with similar influencing factors as for public investments: overall economic development, microeconomic factors of the underlying companies and a fund's internal factors like size, organizational structure or managers' ability. But testing possible explanatory variables on performance, all studies end up with only very low explanatory power. Ljungqvist and Richardson (2003) conducted a very comprehensive analysis on a number of factors from the three categories mentioned above. The adjusted R^2 of their regression was only between 3.7%-5.7% and they only found fund size (negative) and money inflow (negative) to be statistically significant. Kaplan and Schoar (2005, p. 13) conducted a somewhat different analysis and discovered size as a positive driver but again with a low R^2. A negative impact was also attributed to the number of new market entrants. On the other hand, Kaplan and Schoar (2005, p. 21ff) found past performance to be a major driver for future money inflow and the number of new funds which are additionally able to raise more money.

In order to get an idea on the return characteristics of Private Equity investments, it is also interesting to look at the alpha measure which shows the abnormal performance of an investment. For funds, the alpha is often considered as a measure to assess fund managers' abilities. Driessen, Lin and Phalippou (2008) explore risk and return of Private Equity funds in a GMM-framework and also overcome the survivorship bias using aggregated fund data for finished funds that also include unsuccessful investments. They find a large negative alpha of -15% for venture capital and a positive but statistically insignificant alpha for buyout funds independent on fund size. On the one hand, these findings are surprising as Private

Equity is generally seen as a high alpha investment. But it shows again that investors need the ability to find successful fund managers in order to benefit from abnormal performance. Kaplan and Schoar (2005, p. 12) found an average annual alpha of 5% net of fees in their dataset but only on a capital weighted basis. Separating the results for the two main fund types, the alpha for BO funds was -7% and 21% for VC. Cochrane (2005, p. 19) found a median annual alpha of 32% but gross of fees. Gottschalg and Phalippou (2006, p. 18ff) also found gross alpha to be 3% while the alpha net of fees was -3%. Again, this shows the major impact of fees on fund performance and a careful evaluation which general manager is worth its fees. Cvitanic, Lazrak, Martellini and Zapatero (2003a, p. 31) mention three types of risk when estimating the alpha of Private Equity funds: model risk from the underlying model used for the estimation, sample risk from the data analyzed and selection risk from the funds selected.

An interesting study on returns was conducted by Bilo (2002) using data on publicly traded Private Equity investments to estimate returns. The return structure is similar to that reported in chapter 2.2.2. This study is very interesting as the results can be compared to the return characteristics of standard asset classes directly due to the similarities of the data structure and the absence of intermediate valuations by the fund manager influencing the reported NAV. Furthermore, a multi-factor model to explain (publicly traded) Private Equity returns was found to be significant ($R^2 = 63\%$). Influencing factors were stock market volatility, $NASDAQ$ returns, GDP growth, IPO volumes and credit spreads (Bilo, 2002, p. 121ff).

The large differences we see in the literature are somewhat difficult to explain. The main issue is the fact that the different datasets cannot be compared. The differences range from gross- versus net-of-fee data, self-reported versus actual cash flow data or aggregated versus single fund data. However, the probably most important fact is the time period considered. Cochrane for example uses a dataset ending just before the burst of the dotcom-bubble leading to a potential upward bias. This shows the effect of data selection. Hwang et al. (2005) account for the effect of data selection dividing their dataset in two parts, one ending before the dotcom-bubble and the other ending afterwards. The differences are substantial. The investment focus of the several datasets is also often different. While Cochrane uses venture capital data gross of fees, Driessen et al. use net of fee data which also include buyout funds. But Driessen et al. state that fees alone cannot explain those considerable differences. Another major difficulty for comparing the data is the different beta measures used. Measuring betas for Private Equity funds is very difficult as interim fund values have to be included which are prone to some biases as will be shown later. Nevertheless, beta measures avoiding self-reported data will be presented in chapter 2.2.5 to get an understanding of the general risk profile of Private Equity funds.

After this analysis, the return structure of Private Equity investments is still far from being clear and the findings are very sensitive to the dataset, the analyzed time period and the in- or exclusion of fees so that generally holding rules cannot really be inferred. As a result, an investment in Private Equity vehicles can yield extraordinary performance. But as fees are very high and the range of returns extremely large, it is inevitable for an investor to conduct a careful due diligence in order to maximize the probability of picking one of the more successful funds.

2.2.5 Risk Structure of Private Equity Investments

Driessen et al. (2008) calculate the risk profiles for Private Equity funds using a CAPM framework. The findings are very different comparing VC and BO funds. While VC funds have a beta of 3.21, it is only 0.33 for BO funds. Possible reasons for this finding are, that BO funds invest in rather mature (large) and therefore often less risky companies and the more active participation of BO fund managers in the boards helps them to reduce the risks of their investments. The values are significantly positive related to fund size. As a result, the higher returns of larger funds are rather due to higher risk exposure than to abnormal performance as the alpha measure does not depend on size.

On the other hand, Ljungqvist and Richardson (2003) find betas of 1.12 and 1.08 for VC and BO funds respectively. Both studies use the CAPM framework but different datasets. Driessen et al. (2008) use the cash flow data from the *TVE* dataset while Ljungqvist and Richardson use single fund data of a Private Equity investment firm. Driessen et al. estimate alpha and beta from a cross-section of cash flow data at the end of a fund's lifetime using a GMM-framework. As the expected value of discounted investments and payouts must be equal at the end of an investment when all payments are done, the parameters are chosen in a way that matches the observed data. As Ljungqvist and Richardson have access to all intermediate cash flows of a large number of funds, they can estimate the parameters for the CAPM using the valuation and return estimates inferred from those cash flows.

The different datasets might be an explanation of the resulting differences. Another explanation is the fact that Driessen et al. only use mature funds and Ljungqvist and Richardson also include funds in early stages. An implication is to assume that younger funds have lower risk. As these funds are still in the investment phase where basically no cash inflows arise, it is reasonable to assume that the risks only emerge in the later stage where the investments are exited. But trying to explain the differences still leaves the problem of very heterogeneous findings. The BO beta in the first dataset of 0.33 seems extremely low especially when we take the considerable leverage that BO funds typically have into account. As the first dataset includes only a small sample of BO funds and therefore the estimate has a quite large standard deviation, the value of 1.08 from the second dataset can be considered as more realistic but maybe downward biased through the inclusion of early stage funds. As the second dataset is strongly biased towards BO funds, the first estimation of 3.21 for VC fund betas might be more reliable. This is also in line with the findings of Korteweg and Sorensen (2007) who use a Bayesian model to estimate the risks of venture capital. They use investment data from VC funds and include a selection model to the analysis specifying the probability of observing a company's valuation as a function of several variables to account for the endogeneity of valuation events. They find beta values for VC investments ranging from 2.6 to 3.0. What we can infer from the findings is that VC funds are normally riskier than BO funds, in some cases to a very large extent. This result can be found in a vast number of studies on the statistical properties of Private Equity funds. But overall measures of beta for Private Equity funds remain very difficult to compute and therefore have to be interpreted with caution.

Bilo (2002, p. 97ff) finds beta values for publicly traded Private Equity of 0.3 to 1.8. She finds that low beta values are due to low correlation estimates and vice versa. The correlation estimates for publicly traded Private Equity reveal some interesting insights. First, they are only weakly correlated among themselves and with bonds. With different kinds of public equity correlation estimates are between 0.18 and 0.55 being lowest for small-cap shares and the *NASDAQ Composite Index*. The estimated values are especially high for buyout funds investing in large-caps and are completely in line with the correlation estimates Hwang et al. (2005) carried out for their venture capital return index. These findings show that publicly traded Private Equity can serve as a proxy for the statistical properties of Private Equity investments in general.

Phalippou (2007) proposes four explanations for the fact that a growing number of investors put more and more money in Private Equity funds despite the, compared to the risks, low average performance. The first explanation is learning. As general managers of Private Equity funds take part in the operational processes of invested enterprises, the positive effects of their experience on following investment companies are quite obvious. This suggestion is supported by the findings of Connor (2005b, p. 66) and Kaplan and Schoar (2005, p. 15) which show that follow-on funds offer higher returns. Additionally, Cochrane (2005, p. 2) finds follow-on investments to become less risky. But at the same time, Connor (2005b) also discovered that funds from managers which raised only one fund are outperforming funds being established by managers having raised multiple funds. Having in mind the (statistically not significant) outperformance of first-time funds measured by Ljungqvist and Richardson (2003, p. 26), the picture on the persistence of returns becomes even more mixed and considering managers' experience as a good reason to invest in their funds seems rather questionable. As mentioned above, skills to detect funds, which perform better than average, can and should therefore be acquired by investors. The second reason is mispricing. Phallipou mentions that mispricing could arise from different sources; the view on past performance might be biased because of high gross returns, different performance measures across investors, a few successful lighthouse projects, biased data on performance of closed funds and so on. As a third explanation, Phallipou suggests the existence of side benefits which means that investors are not only interested in the performance of the fund but also follow other goals, e.g. the contact to a certain fund manager or personal interest in a target company. The last explanation Phallipou gives is illegal conspiracy.

2.3 Review on Return Measurement

2.3.1 Issues on Return Measurement and Relevant Methods

Some issues about the ways to measure performance have to be considered. There are numerous possibilities to measure the returns of Private Equity investments. When a fund is closed and all investments exited, it is very easy to measure the average annual rate of return. Problems arise mainly from the uncertainty of cash flows when the IRR of a Private Equity fund is computed before its ending date. The IRR is affected by the investment and exit rates of a fund's assets and also by dividends paid which are random ex ante. Private Equity funds are often restricted by covenants to reinvest capital gains (Gompers & Lerner, 1996, p. 481)[14]. As this study examines the optimal use of not invested capital, it will therefore not focus on returned capital further. As it is very difficult to time the cash flows of a Private Equity investment ex ante, comparing IRRs of not yet finished funds is not very meaningful because of the underlying estimations and the influence a fund manager has on the timing of the cash

[14]For a comprehensive study on the use of covenants in Private Equity contracts the reader is referred to Gompers and Lerner (1996).

flows. The IRR is especially sensitive to the terminal value calculation at the exit date. Hwang et al. (2005, p. 26) show that the cash flows from the exit of an investment have a major impact on performance. Furthermore, measurement and reporting of cash flows used to calculate IRRs lie in the fund manager's discretion. Since fund managers have sometimes very different views on the reporting standards of cash flows, reported IRRs are not an ideal measure of a fund's actual return and can therefore vary substantially across funds. Driessen et al. (2008, p. 36) prove this view and show that especially for the group of mature and inactive funds reported NAVs significantly overstate market values. If we try to capture the returns of not yet liquidated funds, the IRR faces obvious shortcomings: the remaining cash flows and the terminal value have to be estimated which stresses the reliability of the resulting IRR. Another point to mention is the fact that Private Equity investors themselves tend to keep funds in the portfolio and report values which are inactive since several years[15]. In the CalPERS portfolio about 5% of the total number of funds are older than twelve years. To address this problem, CalPERS shifted old and inactive funds to an external portfolio manager with the order to liquidate them. As the analysis will not focus on the returns of funds, these problems will not be addressed further. Nevertheless, if the cash flows are modeled carefully, the IRRs from the analysis can be compared and interpreted. The crucial point is to use standardized calculation methods for all cases.

Bader (1996, p. 310ff) shows some different methods to calculate IRRs accounting for the timing of cash flows. A method proposed to address the timing of investments which lies in the fund manager's discretion is the theoretical IRR based on the simultaneous investment of all deals. But as some deals are financed using distributions from other investments this method seems easy but does not capture the deal flow.

As the IRR is not a sufficient measure to judge the return on a Private Equity investment due to the time effect of the capital calls, A. Long and Nickels (1996) developed the so-called Index Comparison Method

[15] Frei and Studer (2004) argue that the payback period is a useful performance indicator in those cases.

(ICM) in order to create a benchmark for Private Equity investments. The main idea behind this method is to track the performance of a Private Equity investment along the performance of a public investment and to compare the results. The procedure is straightforward: each time a private investment experiences a cash in- or outflow, the same cash flow is put in or taken out of an appropriate public market investment, usually an index. It is important to match the timing of the cash flows in and out of the two investments exactly and at the respective prices of the public market investment. At the end of a Private Equity investment the IRRs for both the private and public investments are calculated and can then be compared directly. The crucial idea behind this method is the exact paralleling of the two cash flow streams. If for example two Private Equity investments with the same IRR but different cash flow streams are compared to the same index at the same time, the IRR of the index is expected to be different because of the timing[16]. As a result of this finding, it can be seen that IRR calculation is not sufficient in order to measure performance. The major shortcoming of the Index Comparison Method is the fact that it is only valid over longer time periods in order to incorporate the market cycles. As a typical Private Equity investment has a term of ten years and sometimes longer, the ICM can be considered as fairly reliable. Furthermore, different investments are only compared within the same time periods so that market cycle effects do not influence the performance data. Kaplan and Schoar (2005) use a different measure to evaluate Private Equity performance, the Public Market Equivalent (PME). It calculates the ratio of the present value of future cash flows and the present value of the investment itself compounded at the rates determined by the benchmark index. As A. Long (2008) showed that ICM and PME are equivalent measures and can be translated into each other, this study does not focus further on the PME.

Hwang et al. (2005) developed a Private Equity benchmark index directly. They used single firm funding data and constructed a standardized index

[16] For sample calculations the reader is referred to the appendix in A. Long and Nickels (1996). The effects of different timing are also analyzed in A. Long (2006).

using a hybrid version of the repeated sales technique[17]. This index is a very interesting and helpful tool to benchmark Private Equity investments directly to others and also to compare the overall performance of private investments to indices of public investments. As the focus of this study is not directly related to the comparison of returns of different Private Equity investments, it does not include the index into the following analysis. The index is only calculated for venture capital and gross of fees which have a major impact on performance as was shown in section 2.2.4.

As we have already seen above, IRRs are often a problematic measure of a (Private Equity) fund's performance. This comes from the fact that reported IRRs differ because of simplifying assumptions funds make about cash flows and the special way TVE aggregates them, mixing NAV-growth with cash IRRs. As many studies use the TVE dataset, both shortcomings arise at the same time but the problem does not lie in the general method of the IRR calculation.

2.3.2 Performance Ratios

As IRR measures are biased for several reasons mentioned before, Private Equity Funds often report further performance measures and ratios which are also reported to the TVE database. The goal is to show the characteristics of a fund from different perspectives providing a comprehensive picture. In order to get a common understanding on the quality and the performance of a fund, these measures have to be considered in parallel. The most important are:

$$\text{Distributed-to-paid-in-capital (DPI)} = \frac{\text{Distributed Capital}}{\text{Paid-in Capital}}$$

$$\text{Residual-Value-to-paid-in-capital (RVPI)} = \frac{\text{NAV}}{\text{Paid-in Capital}}$$

[17] This method was first introduced by Bailey, Muth and Nourse (1963) who used sales prices of the same property at different times to overcome quality differences among these properties. A similar version of the model using a different dataset is developed in Hall and Woodward (2003). Using a similar dataset, Peng (2001) also developed a venture capital index combining two sub-indices.

$$\text{Total-Value-to-paid-in-capital (TVPI)} = \frac{\text{Distributed Capital + NAV}}{\text{Paid-in Capital}}$$

$$\text{Distributed-to-committed-capital (DCC)} = \frac{\text{Distributed Capital}}{\text{Committed Capital}}$$

$$\text{Paid-in-to-committed-capital (PICC)} = \frac{\text{Called Capital}}{\text{Committed Capital}}$$

Each of the ratios measures a specific characteristic of a Private Equity investment and can complete the picture when evaluating a fund. There is no discounting considered when calculating the required data. *DPI* measures paid-out capital to invested capital. Depending on the age of a fund, the *DPI* indicates the ability of a fund (manager) to generate a cash back from the investments and will obviously increase over time. Benchmark values for different fund ages can help to compare different funds in the same age class. A common rule-of-thumb is a $DPI=1$ between year 5 to 7 (Bader, 1996). The *DPI* at the end of a fund's lifetime is equal to the investment multiple which is also a frequently reported return measure. The complementary ratio to *DPI* is the *RVPI* which measures the value of unrealized investments still in portfolio (the net asset value, NAV) to invested capital. It is one at the inception of a fund and zero at the ending date. $RVPI>1$ can occur if a revaluation of a fund and a positive expectation on future payoffs increase the NAV. The *TVPI* gives a more general view on total return as it sums up distributions and the value of still active investments. The last two ratios comment on the capital flow into Private Equity investments. These were discussed in detail in chapter 2.2.3. Of course, the major shortcoming of these ratios is the fact that they ignore the time-value of cash flows. Figure 2.11 shows *DPI*, *RVPI* and *TVPI* for VC and BO funds as of June 2010.

Although returns differ between VC and BO funds, the above ratios are very similar on average and only differ somewhat capital weighted showing the time-effect on returns. The finding that larger funds tend to perform worse on average can also be seen from these statistics. As funds with all maturities are included in the data, the average fund age is around 6 years indicating that both types of funds distributed slightly more than

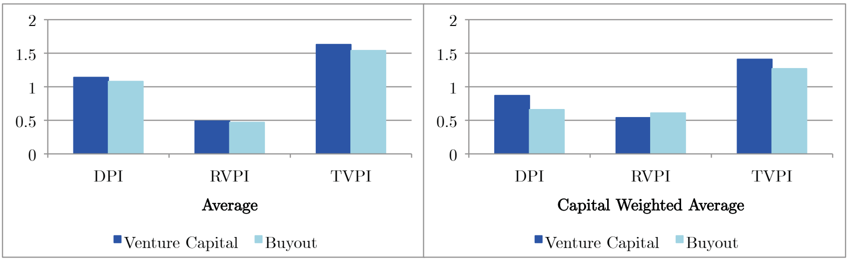


Figure 2.11: Private Equity return ratios (Source: TVE)

the invested capital. The similar figures for VC and BO also show that the major return realization comes at the later stages of a fund. This is supported by the fact that returns are mainly driven by exit returns.

Other return measures commonly reported are pooled returns and size-weighted returns. Pooled returns sum all cash flows over several funds and treat them as if they came from a single fund. This procedure therefore accounts for the timing effect of cash contributions and distributions when calculating average returns over several funds. This is a more accurate way to assess performance than simple averages over fund returns but it does not work for the return calculation of a single fund. Time-weighted returns are calculated using the geometric mean over fixed periods. Cash inflows are assumed to be reinvested. As a result, timing effects are neutralized. But as Bader (1996, p. 314) argues, it is not a meaningful measure to assess Private Equity returns mainly due to the distorting effects of the way the returns are weighted over several periods. As a result, static performance measures can give useful insights on the performance of a fund but can never replace the dynamic procedures mentioned in chapter 2.3.1.

3 Optimal Portfolios

3.1 Introduction to Portfolio Optimization

Portfolio optimization models and the optimal allocation of wealth in order to maximize individual welfare have been an issue in finance since Harry Markowitz published his famous theories on mean-variance-analysis in the Journal of Finance in 1952 (Markowitz, 1952). Since then, portfolio optimization was an active research field in finance focusing on the optimal behavior of individual agents in financial markets. Markowitz' basic idea was to find an optimal balance between risk and return of a portfolio optimizing one factor and holding the other constant. However, at that time, computational requirements were very high especially for computing covariance matrices.

The single index model by Sharpe (1963), modeling absolute returns of risky assets, was a first step to reduce complexity and lead to the famous Capital Asset Pricing Model (CAPM; Sharpe (1964), Lintner (1965), Mossin (1966)) where asset returns are assumed to consist of the risk-free rate plus a market related risk premium which is also very intuitive. As the computational possibilities improved, it appeared that the CAPM had only minor power in predicting asset returns and explaining variability in stock returns. The main problem are correlated residuals. As a result, a lot of effort was undertaken to improve the single-factor model leading to the Arbitrage-Pricing-Theory (APT) developed by Ross (1976). This model further developed the CAPM and allowed expected returns of risky assets to depend linearly on several macroeconomic factors leading to uncorrelated error terms. In the absence of arbitrage, asset prices are assumed to reflect the modeled returns. The main advantage is that the APT-model allows different portfolios with specific factor loadings. Following the work of Ross, research was done to identify the explanatory variables describing asset returns leading to the Fama-French-three-factor-model (Fama & French, 1993). This model states that asset returns not only depend on a market specific risk premium but also on capitalization and book-to-market-ratios. Although it is still a static model with severe shortcomings, it continues to be important in portfolio management till now due to its easiness to use[18].

Tobin (1965a) also introduced real factors influencing the economy to the discussion and extended it to a multi-period problem (Tobin, 1965b). Samuelson (1969) extended the one-period models to multi-period generalizations. Further, he started to think of the optimization problem as a lifetime planning of consumption and investment decisions introducing the utility of consumption. He stuck to a discrete-time version of the model using the dynamic programming method to solve it. This approach was developed by the mathematician Richard Bellman (Bellman, 1957). This method is also mainly used in continuous time solving the *Hamilton-Jacobi-Bellman* partial differential equation resulting from the formulation of the optimality condition. The solution is the resulting value-function. Solving the HJB-equation is subject to the famous *Curse-of-Dimensionality* stating that the computing power increases exponentially with the state variables included. For a comprehensive overview on dynamic optimization and its application on finance the reader is referred to Kamien and Schwartz (1992).

The basic idea of the Markowitz-model is still the same but the assumption of a constant investment opportunity set became a major issue. The fundamental work has been done by Merton (1971) who developed optimization models in continuous time with time-dependent and also stochastic opportunity sets. The continuous time approach has the main advantage of allowing the investor to act immediately to changes in the state of nature[19]. Like Samuelson, Merton also incorporated consumption into

[18] For a comprehensive overview on mathematical theories of risk and return and their application to portfolio optimization strategies the reader is referred to Deng, Wang and Xia (2000).

[19] As this study does not focus on the basic mathematical derivation of optimal investment strategies but only on the enhancements including Private Equity, the reader

the model but found only explicit solutions to very simple cases. Merton (1973) created the consumption CAPM (CCAPM) relating asset returns to the covariance of return and consumption and stating that risky assets can act as a hedge for future consumption uncertainty. Lucas (1978) developed an exchange economy with only one good to study the stochastic behavior of equilibrium asset prices and found a functional equation of asset prices resulting from the optimal utility of consumption over time. These models are the building block for several later studies on asset pricing. The integration of real and financial markets to this problem was done in Cox, Ingersoll and Ross (1985) who developed a continuous time general equilibrium model to examine asset prices which depend on the underlying real variables of the economy. The most famous application is the equity premium puzzle discovered in Mehra and Prescott (1985): the empirical evidence showed that the equity premium requires a coefficient of relative risk aversion of 26.3 (or 17.5 correcting for the bias resulting from return aggregation over time) which was found to be highly unrealistic (Mankiw & Zeldes, 1991, p. 103ff). The puzzle is still not solved completely.

Closed-form solutions to the Merton problem including a stochastic risk-premium can be found e.g. in Kim and Omberg (1996). Evidence on factors influencing asset returns is given in Brennan (1998) and Xia (2001). They also introduced uncertainty on the variables driving these factor dynamics. They conclude that uncertainty about and future learning on mean returns influence portfolio decisions to a large extent. Sørensen (1999) analyzes changes in the opportunity set (stochastic interest rates) and derives strategies to hedge those changes. More complicated versions of the optimization problem with more than one stochastic discount factor influencing the underlying variables of the model are solved in Brennan and Xia (2001) among others. Dondi, Geering, Herzog and Schumann (2004) solve a multifactor-case of the problem using two types of utility functions: constant relative risk aversion (CRRA) and constant absolute

---

is referred to Korn (1997) for a more elementary and comprehensive overview on portfolio optimization techniques under various circumstances. The Merton problem is discussed in detail in many textbooks on continuous-time finance like Bjørk (1998) or Shreve (2004).

risk aversion (CARA). In case of a CARA-function, the optimal weights are a function of current wealth and there exists a finite probability that the investor can go bankrupt. This is not possible with a CRRA-function as the investor rebalances his portfolio over short time periods and can therefore react on losses immediately. Furthermore, Dondi et al. (2004) allow portfolio weights to be restricted. Keel (2006) extents the model for the partial information case where only some of the underlying processes for the factor dynamics are known and others include unknown parameters.

Discrete time versions of the Merton problem are accounted for in detail in Campbell and Viceira (2002). They approximate portfolio returns combining asset log-returns using a Taylor expansion. The approximation is the better the shorter are the time intervals considered. In the limit, this leads of course to Itô's Lemma. Another interesting insight comes from the fact that Campbell and Viceira work with the Epstein-Zin-utility which is a more flexible version of the basic power utility function and allows to separate the relative risk aversion from the intertemporal elasticity of substitution. This provides interesting results for intertemporal long-term portfolio choices. Campbell, Chan and Viceira (2003) solve a discrete version considering multiple factors.

A different method to solve continuous time portfolio optimization problems was developed by Harrison and Pliska (1981), Cox and Huang (1989) and Karatzas (1989), the so-called Martingale Method using general Itô processes as underlyings. The basic idea is the separation of the optimization problem in a *Static Optimization Problem* yielding the optimal consumption process and the terminal wealth of an investor and a *Representation Problem* to find a trading strategy leading to those optima. The Martingale Measure itself serves as a stochastic discount factor and is unique if markets are complete. The major advantage is the fact that the two problems can be solved separately. Today's research is especially focussed on relaxing the model assumptions. Korn (1997) for example allows for transaction cost, Munk and Sørensen (2004) allow for stochastic interest rates.

3.2 The Merton Portfolio

3.2.1 Derivation of the Merton Portfolio

The importance of the Merton portfolio in finance is extremely far-reaching as it can be seen not only as a benchmark portfolio in financial modeling and portfolio optimization but the Growth Optimal Portfolio (GOP) as a special case can also be used as a numéraire in contingent claims pricing and risk management[20].

The Growth Optimal Portfolio can be defined as the portfolio that maximizes the expected logarithmic utility from consumption and final wealth by optimizing the consumption path and the share of wealth invested in the risky asset, i.e. it is the portfolio that outperforms the median growth rate of any other portfolio in the long-run. The market portfolio can always be described as a combination of the GOP and the risk-free bond. Originally, the idea was developed in the context of winning strategies in lotteries. The first who applied the theory explicitly to "financial investments" was Breiman (1961) showing that there is an asymptotically optimal strategy maximizing the initial bet. Since then, financial theory has gone a long way.

In this section a solution to the Merton portfolio is derived using the Martingale Method. First, we have to formulate some assumptions in order to apply the method. The most important assumption is the one of complete markets which means that all contingent claims are marketed, i.e. all assets are traded and the number of stochastic variables equals the number of risky assets. Further assumptions are the self-financing condition, allowance for short-selling, exclusion of transaction costs, freely divisible assets and an infinite number of investors[21]. As the goal is to maximize expected utility, we have to define a utility function which has to fulfill several conditions: twice differentiable over the relevant range,

[20] For detailed derivations of the numéraire portfolio the reader is referred to J. Long (1990).

[21] For a more detailed discussion on the assumptions the reader is referred to any standard textbook on continuous time portfolio optimization, e.g. Korn (1997).

strictly monotonic and concave and the INADA-conditions. A specific utility function is not formulated, yet.

The formulation of the problem is

$$\max_{\pi,c} E \int_0^T u_1(t, c(t))dt + u_2(B) \tag{3.1}$$

with π being the share invested in the risky asset and $c(t)$ being the consumption path. B is final wealth and $u_1(.)$ and $u_2(.)$ the utility functions. This problem can be split in a *Static Optimization Problem (SOP)* and a *Representation Problem (RP)*.

Assuming $u = u_1 = u_2$ we have

$$\begin{aligned} &\max_{B,c} E \int_0^T u(c(s))ds + u(B) && (SOP) \\ &s.t. E \int_0^T (H(s)c(s)ds + H(T)B) \leq X_0 \end{aligned} \tag{3.2}$$

$$X^{X_0,\pi^*,c^*}(T) = B^* \qquad (RP) \tag{3.3}$$

X_0 is initial wealth by which expected consumption and terminal wealth have to be bounded as we assume self-financing. π^* and c^* are the optimal investment and consumption path and B^* is the respective final wealth solving the *SOP*. $H(t)$ is the State-Price-Deflator or Pricing Kernel which is a universal measure to discount future payoffs into today's prices. It is developed throughout the following lines. Asset prices follow

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dZ_{S,t} \tag{3.4}$$

$$\frac{dB_t}{B_t} = rdt \tag{3.5}$$

with μ_S and σ_S being expected return and standard deviaton of a standard Brownian motion and r being the risk-free rate. $Z_{S,t}$ is a standard-normally distributed random variable. Because of the *Girsanov Theorem*, we can change the drift term of a given Itô process without changing the diffusion term. As a result, we can formulate a new (risk-neutral) probability measure for the risky asset S resulting in a drift term equal to the risk-free rate. The solution to equation 3.4 therefore is:

$$f(S_t) = \frac{1}{S_0 \sigma \sqrt{t}} e^{-\frac{1}{2\sigma^2 t}(ln(\frac{S_t}{S_0}) - rt)^2} \tag{3.6}$$

The advantage of this risk-neutral probability measure compared to the statistical measure is captured by the famous *Fundamental Asset Pricing Theorem* of Harrison & Pliska:

> A financial market that has a risk-neutral probability measure admits no arbitrage. If every contingent claim is priced the risk-neutral measure is unique.

After the transformation, we are able to calculate the *Radon-Nikodym-Derivative* $h(t)$ dividing the statistical and the risk-neutral probability measure. Using the fact that risky returns are lognormally distributed we have

$$h(t) = e^{-\frac{(\mu_S - r)^2}{2\sigma^2} t - \frac{\mu_S - r}{\sigma} B_t} \tag{3.7}$$

Using integration rules and replacing the expression for the Sharpe ratio by θ leads to

$$h(t) = e^{-\int_0^t \theta(s) dB_s - \frac{1}{2} \int_0^t \|\theta(s)\|^2 ds} \tag{3.8}$$

Assuming that the *Novikov Condition* holds is sufficient for $h(t)$ to be a Martingale[22]. This condition states that the expectation of the second integral of the optimization problem is finite. Multiplying the *Radon-Nikodym-Derivative* with a discounting process, we get the pricing kernel $H(t)$

$$H(t) = e^{-\int_0^t (r + \frac{1}{2}\|\theta(s)\|^2) ds - \int_0^t \theta(s) dB_s} \tag{3.9}$$

The *SOP* is solved using a regular Lagrangian and the *RP* by comparing coefficients of the underlying Brownian motion and the general wealth

[22] For a proof the reader is referred to Karatzas and Shreve (1991, 198-201).

process.

If we assume the utility function to be of power utility type with $\gamma > 1$, a subclass of the HARA-type functions, we get a closed form solution which is also related to the coefficient of risk aversion having a constant relative risk aversion of $\gamma > 1$ called the Merton portfolio. For $\gamma \rightarrow 1$ the utility function is of log-type which leads to the Growth Optimum Portfolio. The following derivations are therefore based on that specific type of utility function.

For the optimal process $X^*(t)$ we get

$$X^*(t) = \frac{1}{H(t)} E(H(T)B^*|F_t) \tag{3.10}$$

Assuming constant market coefficients, $H(t)$ can be simplified to

$$H(t) = e^{-rt-\theta B_t - \frac{1}{2}\theta^2 t} \tag{3.11}$$

Solving this problem using the methodology described above, we end up with an optimum inversely related to the coefficient of risk aversion, the famous Merton portfolio[23]:

$$\pi_M^* = \frac{\mu_S - r}{\gamma \sigma^2} \tag{3.12}$$

For $\gamma = 1$, 3.12 can be shown to be the GOP and the utility function converges to a log-function. This is optimal to any other trading strategy in the long-run which is shown in the following section.

3.2.2 Optimality of the Growth Optimal Portfolio

It can be shown that no portfolio can generate wealth over an increasing time horizon that is strictly larger than that from the GOP with some strictly positive probability[24]. Samuelson (1969) already showed that maximizing expected utility for isoelastic functions (i.e. the log-

[23] For a detailed derivation of the solution and further problems of the same type see Munk (2008).

[24] See i.e. Platen (2005) or Korn and Schäl (1999) for detailed derivations.

function) results in constant investment weights over time maximizing wealth and consumption compared to any other trading strategy. Platen (2005, p. 384ff) shows that the growth rate of the GOP is at no point in time smaller than the growth rate of any other portfolio and that the trajectory of the GOP is always path-wise equal or larger than of any other portfolio after a sufficiently long time. When the GOP is used as a benchmark portfolio, its conditional expected return is zero and there is no other benchmarked portfolio generating expected returns larger than zero (J. Long, 1990).

Systematically outperforming the GOP is not possible with some strictly positive probability. Baumann and Müller (2006) show that beating the GOP is even for very low rates of outperformance connected to an extremely high shortfall probability compared to a targeted rate of outperformance. They calculate the shortfall probability for different time horizons and define the shortfall for different degrees of success, saying that a shortfall only occurs if the degree of outperformance of a targeted wealth level is lower than a fraction “a” of the GOP-level of wealth with $a \in (0, 1]$. Two types of shortfall are defined to get an understanding of the risk. Type I defines the shortfall in case the investment is lower than a fraction “a” of the GOP wealth at any point of time in the considered interval [0,T] and Type II defines the shortfall if the investment is lower than than a fraction “a” of the GOP wealth at time T, the investment horizon. It is obvious that a shortfall of Type II is much less probable and leads to a more risky investment path. The following graphs show the minimal shortfall probabilities for both types with various target rates of outperformance c. The GOP-rate of return is assumed to be 9.25% with a risk premium of 6.25% and a standard deviation of 25%. The wealth processes follow a geometric Brownian motion.

Type I is independent of T. The shortfall probability is extremely high being above 50% for $a = 0.9$ and $a = 0.8$ at rates of outperformance as low as 0.7%. In case of $a = 0.6$, the shortfall probability only increases very slowly being at 6.89% for $c = 0.5\%$. Of course, the buffer is extremely high in this case. From figure 3.2 it can be shown that the shortfall probability

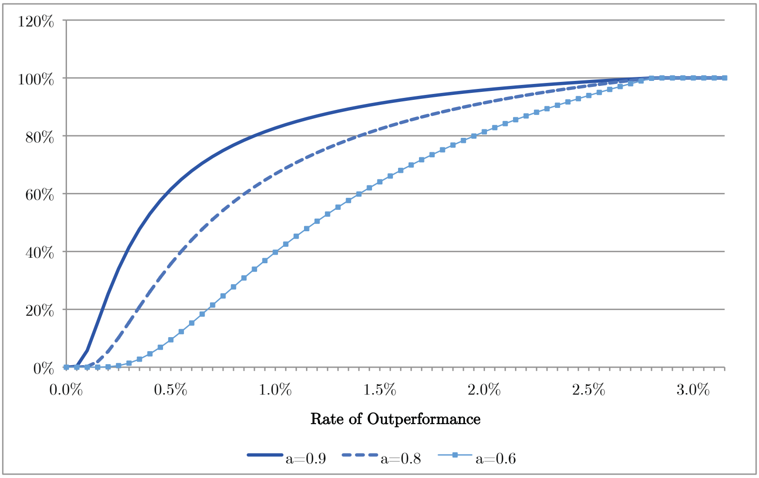


Figure 3.1: Shortfall probability of type I (Source: Adapted from Baumann and Müller (2006))

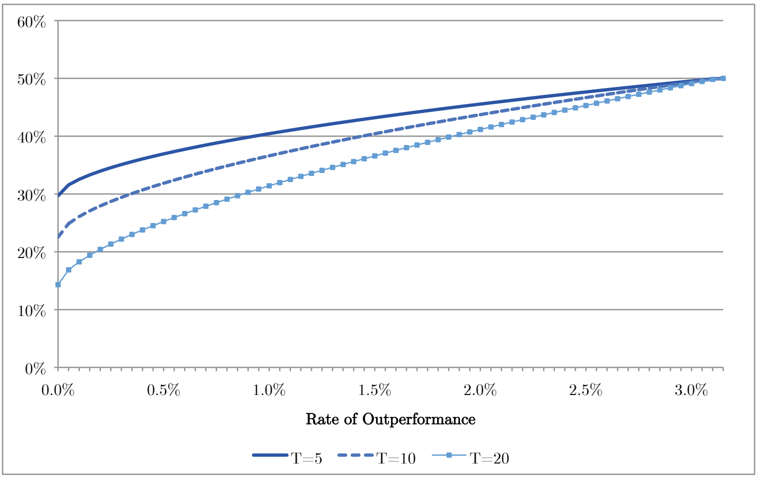


Figure 3.2: Shortfall probability of type II ($a = 1$) (Source: Adapted from Baumann and Müller (2006))

of Type II is always below 50% and strongly depends on the time horizon T considered for the outperformance. But even for a rather long time horizon ($T = 20$) and a moderate outperformance of 0.25% the shortfall probability is above 20%.

3.3 Portfolio Optimization including Private Equity

3.3.1 Static Optimization Methods

One of the earliest studies on the allocation of funds to different asset classes based on the return premium of a single asset was conducted by Treynor and Black (1973) stating that the optimal weights only depend on the specific structure of the asset and not on the risk-return profile of the market resulting in the appraisal ratio which is the risk adjusted outperformance of an investment measured by the CAPM α. The underlying model was based on the covariances of the assets and was build on the foundation of Markowitz (1952) and Sharpe (1966). Black and Litterman (1992) refined the setup allowing for independent and unknown shocks modeling the excess returns of the assets. One of the earliest studies to calculate minimum-variance portfolios including alternative investments (hedge funds) is Amenc and Martellini (2002). Optimal portfolio weights are calculated from covariance estimates of hedge fund indices. An extension to the standard mean-variance optimization framework is provided in King (2007) who accounts for the fact of over-concentrated optimal portfolios that are not feasible for institutional investors due to investment guidelines.

In order to determine the periodic risk of Private Equity funds, A. Long (1999) uses the spread of portfolio terminal values and calculates the risk required to generate them backward. This procedure ensures that risk is calculated without having to rely on biased intermediate valuations. Returns are also estimated from a cross-section of final fund values. As a result, A. Long (1999) derives optimal portfolio weights using a mean-variance analysis for different correlation assumptions. The most interesting finding is the fact that the weight of Private Equity is only marginally sensitive to a change in the correlation structure and lies around 30% for the maximum Sharpe ratio portfolio. Baierl, Chen and Kaplan (2002) use the assumption of lognormally distributed returns to build a model for the return and volatility estimation for venture capital funds in a maximum likelihood framework. Corrected for standard deviation they do not find outperformance of VC investments compared to neither small nor large stocks but in a mean-variance framework Private Equity was always included in the range between 2% - 9% due to its low correlation. Although these results are very different mainly due to the model specifications, they show that the inclusion of Private Equity optimizes the risk-return structure of a portfolio significantly.

Kerins, Smith and Smith (2004) estimate returns using the CAPM framework and derive the related opportunity cost of capital for different Private Equity investors. They consider two types of investors: entrepreneurs (Private Equity as a single investment) and financial investors (Private Equity as part of a well-diversified portfolio). They found out that the opportunity costs of the investment are two to four times higher for an underdiversified entrepreneur. These costs increase with the illiquidity of the asset and the degree of underdiversification. They also tested various levels of underdiversification and found the results to be very robust.

3.3.2 Stochastic Optimization Methods

The obvious problem is the static nature of the optimization frameworks described in the previous section. As a result, starting with the work of Merton, stochastic optimization methods were developed with some recent versions also including alternative investments to the investment opportunity set. An interesting comparison of static and stochastic portfolio optimization methods is done in Lenoir and Tuchschmid (2001). They compare the results from a standard mean-variance approach to the Merton model in a practical application and mainly focus on the effect of the investment time horizon but do not include alternative assets. Longstaff

(2001) proposed a continuous time portfolio optimization model accounting for the illiquid nature of securities. He assumed that an investor faces trading constraints and as a result is forced to invest in a suboptimal portfolio loosing welfare gains compared to the case when assets are completely liquid. Døskeland (2007) developed an asset allocation framework similar to Campbell and Viceira (2002) taking into account an investor's balance sheet to evaluate the long-term relationship between non-tradable assets, stocks and bonds. Considering the long-term and illiquid nature of those assets he found the optimal asset allocation to change over time and to depend strongly on the long-term relationship of the assets. One of the earliest studies incorporating alternative assets directly into a continuous time framework is the one of Cvitanic et al. (2003a; 2003b) which derives closed form solutions for the optimal investment path. The idea of the CAPM-framework applied is similar to Baks, Metrick and Wachter (2001) who introduce prior beliefs about the CAPM α which is consistent with the fund manager's abilities. The investment opportunities in Cvitanic et al. (2003a) consist of a risk free bond, a public market index and an alternative asset. The dynamics of the public market index follow the common stochastic differential equation with lognormal processes and for the alternative asset the stochastic differential equation is embedded in a CAPM like framework[25] which defines the expected return as the risk adjusted return with respect to the market premium. The diffusion terms are assumed to be independent but the alternative asset also depends on the diffusion of the public market index. The goal of the analysis is to maximize the expected utility from final wealth using a constant relative risk aversion (CRRA) utility function. For the processes of alternative, public and risk-free assets we have

$$\frac{dA_t}{A_t} = \mu_A dt + \sigma_A dZ_{A,t} + \rho\sigma_S dZ_{S,t}$$

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dZ_{S,t} \tag{3.13}$$

[25] It does not require the CAPM to hold because only its terminology is used. This has the advantage that the results can be related to α and β measures which are commonly used to describe Private Equity funds. The generality of this approach is shown in Cvitanic et al. (2003b), chapter 5.

$$\frac{dB_t}{B_t} = rdt$$

with μ and σ being expected return and standard deviation of the alternative (A) and public (S) assets and r being the risk-less return. Z_t is the respective standard-normally distributed random variable. As the expected return of the alternative asset is expressed in a CAPM-framework we have $\mu_A = r + \beta(\mu_S - r) + \alpha$ with $\beta = \rho\frac{\sigma_A}{\sigma_S}$. The utility function used is of CRRA-type with

$$u(W) = \frac{1}{1-\gamma}W^{1-\gamma} \tag{3.14}$$

and γ being the individual coefficient of relative risk aversion. Cvitanic et al. (2003a) assume that neither expected returns nor stochastic parts are observable. The investor only observes the price processes (A_t, S_t) and therefore the information only consists in the $\mathbb{P}$-augmentation of the filtration $\mathcal{F}_t := \sigma(A(s), S(s); 0 \leqslant s \leqslant t)$ generated by the two price processes. Risk premia are defined by

$$\begin{aligned} \theta_S &= \frac{\mu_S - r}{\sigma_S} = SR_S \\ \theta_A &= \frac{\mu_A - r}{\sigma_A} - \rho\frac{(\mu_S - r)}{\sigma_S} = SR_A - \rho SR_S \end{aligned} \tag{3.15}$$

with SR being the respective Sharpe ratios. It is assumed that the vector $\theta = (\theta_S, \theta_A)$ has a normal prior distribution:

$$\theta \sim N(\phi = \begin{pmatrix} \phi_S \\ \phi_A \end{pmatrix}, \Delta = \begin{pmatrix} \delta_S & 0 \\ 0 & \delta_A \end{pmatrix}) \tag{3.16}$$

Priors are uncorrelated in this case[26]. The resulting optimal investment paths for the alternative asset (x^*) and the public market investment (s^*) yield very interesting implications[27]:

$$x^* = \frac{\alpha}{\sigma_A^2(\gamma - (1-\gamma)\delta_A T)} \tag{3.17}$$

[26]For correlated priors the reader is referred to Cvitanic et al. (2003b, p. 14ff).
[27]For a proof the reader is referred to Cvitanic et al. (2003b).

$$s^* = \frac{\mu_S - r}{\sigma_S^2(\gamma - (1-\gamma)\delta_S T)} - \beta x^* \tag{3.18}$$

α and β are the respective parameters from the CAPM specification and δ refers to the variance of the priors on risk-premia. For a solution to exist, $\gamma > 1$ has to hold.

The optimal amount of money invested in alternative assets directly depends on the abnormal performance α. As a result, investments in Private Equity do only make sense if abnormal performance is expected to be greater than zero. The greater the uncertainty around α the smaller the optimal amount of alternative investments, which can be inferred from the negative impact of δ_A, the variance of the prior on the risk-premium. Only in cases when the variance of the prior on the risk-premium is not zero, the optimal investment path is time-dependent and negatively impacted by longer time-horizons as the adverse effect of the uncertainty about expected returns is the larger the longer the investment horizon. If $\delta = 0$, the investor has perfect information on the underlying process determining μ_S and we end up with the famous Merton solution.

For the public share the findings are more interesting. In addition to the factors also driving the investment path of the alternative asset, the optimal amount invested in the public market negatively depends on the investment in the alternative asset weighted by β, the investment-specific risk. Cvitanic et al. (2003b, p. 11) show that the sum of risky investments for $\beta < 1$ is larger than in cases without alternative investments showing the diversification effect of alternative investments. As the investment opportunity set is constant, we do not have the intertemporal hedging component in either of the two cases. Using the definition of β, we can modify the results somewhat:

$$\beta = \rho\frac{\sigma_A \sigma_S}{\sigma_S^2} = \rho\frac{\sigma_A}{\sigma_S} \tag{3.19}$$

$$s^* = \frac{\mu_S - r}{\sigma_S^2(\gamma - (1-\gamma)\delta_S T)} - \rho\frac{\sigma_A}{\sigma_S}x^* \tag{3.20}$$

For large absolute values of correlation, the asset allocation becomes the most extreme. In case of no correlation and no uncertainty ($\delta_S = 0$), s is the same as in the standard Merton case and money invested in the alternative asset is the least and comes at the expense of the risk-free investment. The modified solution for s derived above is exactly the same found by Keel (2006, p. 89) who conducts a similar analysis comparing cases with full and partial information on the priors. For x, he discovers some further interesting findings. The higher the absolute value of the correlation between public and private asset the higher is the optimal weight of the private asset. In case of negative correlation, the optimal public market weight is larger than in the positively correlated case and comes at the expense of the risk-free investment as x depends quadratically on ρ. This comes from the fact that for lower correlation downswings in the private asset are better diversified by the public market investment.

A very interesting finding of Cvitanic et al. (2003a, p. 31) is the fact that for $\beta < 0.5$ the alternative asset serves as a substitute of the risk-free investment as in the optimum more money is withdrawn from the risk-free investment than from the public market investment when alternative assets are taken into account. In their base case[28] about 28.64% is invested in the alternative asset on average of which 27.64% was redirected from the risk-free account and only 1% from the public market account. This shows the extremely low β-estimate in their dataset. The absolute values have to be interpreted with great caution as they are extremely sensitive to the underlying assumptions, the time period considered and the dataset used.

3.3.3 Modeling Portfolio Optimization including Private Equity

In this section, a portfolio optimization problem will be solved using the underlying model characteristics that will also be used for later analysis. The major shortcoming of the model shown in section 3.3.2 is the CAPM-

[28] View Cvitanic et al. (2003a, p. 35) for exact parameter values.

like structure. Due to the challenges when estimating α it is difficult to evaluate portfolio weights. As was shown above, return estimates from publicly traded Private Equity are much more suitable also for illiquid Private Equity (see Bilo (2002)). The model is similar to the continuous time approach derived in Denzler, Müller and Scherer (2001). The asset classes comprise a Private Equity fund, a risky asset (i.e. a stock index) and a risk-free bond as an equivalent to liquidity. Private Equity investments are assumed to follow a standard stochastic process, as well as the risky (public) asset. The corresponding differential equations for the underlying price processes are

$$\frac{dC_t}{C_t} = \mu_P dt + \sigma_P dZ_{P,t} \tag{3.21}$$

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dZ_{S,t} \tag{3.22}$$

$$\frac{dB_t}{B_t} = r dt \tag{3.23}$$

with μ_P, μ_S and r being the corresponding returns, σ_P and σ_S the standard deviation of the Private Equity fund and the risky asset and Z_t being the respective standard Brownian motions with a correlation of ρ between private and public equity. The general assumption found in the literature is used here setting invested capital equal to committed capital in order to define the benchmark case to which the results derived in later chapters can be compared. This allows to see the effects of the investment delay which is the focus of this analysis. Furthermore, the standard assumptions of similar frameworks are made: the markets are complete and trading is continuous, no transaction costs, the investor is a price taker and has no market power to influence asset prices. The portfolio weights of the respective asset classes are defined as:

$$x = \frac{C_t}{W_t} = \frac{C_t}{C_t + S_t + B_t} \tag{3.24}$$

$$s = \frac{S_t}{W_t} \tag{3.25}$$

$$b = \frac{B_t}{W_t} = 1 - x - s \tag{3.26}$$

summing up to one. Furthermore, the optimal portfolio weights can be shown to be constant under the given conditions (geometric brownian motion and constant relative risk aversion). A detailed proof is given in Cvitanic and Karatzas (1992, p. 767-780) which not only includes unconstrained problems but also introduces some types of constraints (incomplete markets, short-selling constraints and different rates for borrowing and lending). As a result, the problem of maximizing the expected utility of final wealth can be reduced to a static optimization problem. Denzler et al. (2001) also show that optimal weights are constant introducing an asset liability framework to overcome the shortcomings of the basic (constrained and unconstrained) model with very short investment horizon and make it applicable for investors with longer investment horizons (especially pension funds). The problems mainly arise from estimating the covariance structure of the risky assets which can only be done in a straightforward way when there are less than 20 asset classes. To overcome these problems, the following analysis only assumes 3 asset classes. The expected return of Private Equity, μ_P, can for example be estimated from historical returns of listed (and therefore liquid) Private Equity vehicles or from the aggregated TVE return index. The difficulties with the TVE dataset are described in section 2.2.2.

As the investment period is assumed to be $T - t_0$ (with $t_0 = 0$), final log-wealth can be defined as $lnW_T = ln(C_T + S_T + B_T)$. Using the dynamics from equations 3.21 - 3.23 and Itô's Lemma the dynamics of $\mathrm{d}ln(W_t)$ can be derived:

$$\begin{aligned}
\mathrm{d}ln(W_t) = {} & \frac{1}{C_T + S_T + B_T} dC_t + \frac{1}{C_T + S_T + B_T} dS_t + \frac{1}{C_T + S_T + B_T} dB_t \\
& - \frac{1}{2} \frac{1}{(C_T + S_T + B_T)^2} (dC_t)^2 - \frac{1}{2} \frac{1}{(C_T + S_T + B_T)^2} (dS_t)^2 \\
& - \frac{1}{2} \frac{1}{(C_T + S_T + B_T)^2} (dB_t)^2 - \frac{1}{2} \frac{2}{(C_T + S_T + B_T)^2} dC_t dS_t
\end{aligned} \tag{3.27}$$

Due to Itô calculus, terms of higher order in dt are set equal to 0. As dB_t does not depend on a stochastic part, $(dB_t)^2$ can also be set to 0. After canceling out and introducing the corresponding portfolio weights we get[29]

$$\mathrm{d}ln(W_t) = x\frac{dC_t}{C_t} + s\frac{dS_t}{S_t} + b\frac{dB_t}{B_t} - \frac{1}{2}x^2(\frac{dC_t}{C_t})^2 - \frac{1}{2}s^2(\frac{dS_t}{S_t})^2 - xs\frac{dC_t}{C_t}\frac{dS_t}{S_t} \tag{3.28}$$

Replacing the dynamics with the respective differential equations gives the stochastic differential equation for log-wealth:

$$\begin{aligned}\mathrm{d}ln(W_t) = {} & [x\mu_P + s\mu_S + (1-x-s)r - \frac{1}{2}x^2\sigma_P^2 - \frac{1}{2}s^2\sigma_S^2 - xs\sigma_P\sigma_S\rho]dt \\ & + x\sigma_P dZ_{P,t} + s\sigma_S dZ_{S,t}\end{aligned} \tag{3.29}$$

From equation 3.29 the expression for final log-wealth is straightforward (for $W_0 = 1$):

$$\begin{aligned}ln(W_T) = {} & [x\mu_P + s\mu_S + (1-x-s)r - \frac{1}{2}x^2\sigma_P^2 - \frac{1}{2}s^2\sigma_S^2 - xs\sigma_P\sigma_S\rho]T \\ & + x\sigma_P Z_{P,T} + s\sigma_S Z_{S,T}\end{aligned} \tag{3.30}$$

Maximizing the expected utility of final wealth yields the optimal portfolio weights across the three defined asset classes. The utility function $u(.)$ used is of CRRA-type (see equation 3.14). This leads to

$$\begin{aligned}E[u(W_T)] = {} & \frac{1}{1-\gamma}e^{(1-\gamma)(x\mu_P + s\mu_S + (1-x-s)r - \frac{1}{2}x^2\sigma_P^2 - \frac{1}{2}s^2\sigma_S^2 - xs\sigma_P\sigma_S\rho)T} \\ & E[e^{(1-\gamma)(x\sigma_P Z_{P,T} + s\sigma_S Z_{S,T})}]\end{aligned} \tag{3.31}$$

[29] For detailed derivations the reader is referred to Appendix C.1.

Calculating the expectation[30] of the stochastic part results in

$$E[u(W_T)] = \frac{1}{1-\gamma}\exp[(1-\gamma)(x\mu_P + s\mu_S + (1-x-s)r - \frac{1}{2}x^2\sigma_P^2 - \frac{1}{2}s^2\sigma_S^2 - xs\sigma_P\sigma_S\rho)T + \frac{1}{2}(1-\gamma)^2(x^2\sigma_P^2 T + s^2\sigma_S^2 T + 2xs\sigma_P\sigma_S\rho T)] \tag{3.32}$$

Maximizing the expected utility of final wealth can be reduced to maximizing the exponent of equation 3.32 divided by $(1-\gamma)T$. If we introduce $F(x,s)$ as the respective function to be optimized, the problem is:

$$\max_{x,s}\ E[u(W_T)] \hat{=} \max_{x,s}\ F(x,s) \tag{3.33}$$

with

$$F(x,s) = x\mu_P + s\mu_S + (1-x-s)r - \frac{1}{2}\gamma x^2\sigma_P^2 - \frac{1}{2}\gamma s^2\sigma_S^2 - \gamma xs\sigma_P\sigma_S\rho$$

The first order conditions (FOC) are then straightforward to derive:

$$\frac{\delta F}{\delta x} = \mu_P - r - \gamma x\sigma_P^2 - \gamma s\sigma_P\sigma_S\rho = 0 \tag{3.34}$$

$$\frac{\delta F}{\delta s} = \mu_S - r - \gamma s\sigma_S^2 - \gamma x\sigma_P\sigma_S\rho = 0 \tag{3.35}$$

Using matrix notation the system can be simplified to

$$\gamma V\boldsymbol{w} = \boldsymbol{\mu} \tag{3.36}$$

with V being the covariance matrix, $\boldsymbol{w}$ being the vector of optimal portfolio weights and $\boldsymbol{\mu}$ being the vector of the respective risk premia:

$$V = \begin{pmatrix} \sigma_P^2 & \sigma_P\sigma_S\rho \\ \sigma_P\sigma_S\rho & \sigma_S^2 \end{pmatrix};\boldsymbol{w} = \begin{pmatrix} x \\ s \end{pmatrix};\boldsymbol{\mu} = \begin{pmatrix} \mu_P - r \\ \mu_S - r \end{pmatrix} \tag{3.37}$$

[30] For detailed derivations the reader is referred to Appendix C.2.

The optimal weights therefore are

$$\boldsymbol{w}^* = \frac{1}{\gamma} V^{-1} \boldsymbol{\mu} \tag{3.38}$$

which corresponds to the famous Merton solution. It is obvious that the optimization problem has only a unique solution if the determinant of the covariance matrix V is not equal to zero otherwise the inverse of V does not exist. Therefore we need the condition $-1 < \rho < 1$ to hold. Since the covariance matrix is positive definite, the solution is a global maximum. Using Cramer's rule results in

$$\boldsymbol{w}^* = \frac{1}{\gamma \det(V)} \begin{pmatrix} \sigma_S^2 & -\sigma_P \sigma_S \rho \\ -\sigma_P \sigma_S \rho & \sigma_P^2 \end{pmatrix} \boldsymbol{\mu} \tag{3.39}$$

and expressing the respective weights separately and rearranging yields the optimal portfolio weights that maximize an investor's utility of final wealth. From x^*_{NG} and s^*_{NG} one can directly derive b^*_{NG}:

$$x^*_{NG} = \frac{\mu_P - r - \frac{\sigma_P}{\sigma_S}\rho(\mu_S - r)}{\gamma \sigma_P^2 (1-\rho^2)} = \frac{SR_P - SR_S \rho}{\gamma \sigma_P (1-\rho^2)} \tag{3.40}$$

$$s^*_{NG} = \frac{\mu_S - r}{\gamma \sigma_S^2} - \frac{\sigma_P}{\sigma_S} \rho x^*_{NG} = \pi_M - \frac{\sigma_P}{\sigma_S} \rho x^*_{NG} \tag{3.41}$$

$$b^*_{NG} = 1 - x^* - s^* = 1 - \pi_M - (1 - \frac{\sigma_P}{\sigma_S}\rho) x^*_{NG} \tag{3.42}$$

with SR being the respective Sharpe Ratios of private and public equity. The index NG indicates that the time-lag of investments into the Private Equity fund is not considered. The Private Equity investment depends on the Sharpe Ratio of private and public equity, the correlation between private and public assets and an investor's risk aversion. The Private Equity investment reduces public equity and risk-free bonds. The correlation structure of private and public equity reflected by ρ has interesting implications on the optimal portfolio weights. For negative correlation ($\rho < 0$), the optimal share of public equity is larger than in the standard Merton case. If there is no correlation ($\rho = 0$), private and public equity are invested along the Merton portfolio rule and Private Equity investments come fully at the expense of risk-free bonds. In this context, Private Equity investments can be seen as a perfect substitute for liquidity which is a very interesting finding. A reasonable explication might be the fact that downturns in the private asset are compensated by public equity in case of no (or very low) correlation and no hedge is needed. This is equivalent to the results of Cvitanic et al. (2003a, p. 31) and can be seen from the optimal weights directly. A formal proof is given in Cvitanic et al. (2003b, p. 11ff). If the correlation is equal to the ratio of risk ($\rho = \frac{\sigma_S}{\sigma_P}$), Private Equity comes at the expense of the risky public asset. An explanation are again hedging considerations. For larger levels of correlation public equity is reduced at an even larger proportion than the investment in private equity.

Constraints to the investment weights can easily be implemented in this model (i.e. no short-selling, currency hedging, β-neutral strategies etc.) as is shown in Denzler et al. (2001, p. 23) or Cvitanic et al. (2003b, p. 23).

The dynamics of the optimal weight of Private Equity is obvious for the Sharpe ratios and the risk aversion. As equation 3.40 depends quadratically on ρ, the sensitivity of the optimal share of Private Equity in the portfolio on correlation is not straightforward. It depends on the respective Sharpe ratios of private and public equity. For very low levels of negative and very high levels of positive correlation, it might be beneficial for an investor to reduce Private Equity as the correlation falls/increases further as diversification benefits tend to decrease.

4 Modeling Optimal Private Equity Investments

As we have seen in the discussion of the previous chapter, studies on the portfolio optimization including Private Equity do not take the delayed investment path into account. As Private Equity is very often only a small proportion in a large investor's portfolio[31], many investors assume to be able to pay called commitments out of the available liquid funds. Obviously, such a behavior is ex ante not optimal from the perspective of a comprehensive portfolio optimization approach. The following chapter establishes a model to optimally allocate the not (yet) invested capital on public equity and bonds. This will be done in two main parts. We consider a pure Private Equity investor. First, a model is developed capturing the opportunity cost from delayed investment. In a second step, a rule to optimally allocate the capital between stocks and bonds is derived analytically. In chapter 5, we consider Private Equity as a part of a larger portfolio consisting of Private Equity, stocks and bonds.

4.1 Model Setting

Assume that we have a Private Equity fund, a risky asset (i.e. a stock index) and a risk-free bond. The corresponding differential equations for the price processes are again

$$\frac{dC_t}{C_t} = \mu_P dt + \sigma_P dZ_{P,t} \tag{4.1}$$

[31] Baierl, Chen & Kaplan (2002) estimate a number of 5% over total invested assets, which are invested in Private Equity.

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dZ_{S,t} \tag{4.2}$$

$$\frac{dB_t}{B_t} = rdt \tag{4.3}$$

with μ_P, μ_S and r being the corresponding returns, σ_P and σ_S the standard deviation of the Private Equity fund and the risky asset and Z_t being the respective standard Brownian motions with a correlation of ρ. The solution to the first two differential equations is well known with

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma Z_t} \tag{4.4}$$

The risky returns are therefore distributed lognormally following

$$ln(\frac{S_t}{S_0}) \sim N((\mu - \frac{1}{2}\sigma^2)t, \sigma\sqrt{t}) \tag{4.5}$$

The investor wants to invest his entire initial wealth in the Private Equity investment. In order to simplify the analysis, it is assumed that an investor can pay in an amount δ of his initial commitment C_0 to the fund at time $t_0 = 0$ and the remainder at time t_1. C_0 is set to one without loss of generality. Therefore, we can split the analysis in two time periods $\tau_1 = [0, t_1]$ and $\tau_2 = [t_1, T]$. The interesting question is how to optimally allocate $(1-\delta)$ of the committed capital in period τ_1 to stocks and bonds[32]. The wealth processes in the two periods follow

$$t \in \tau_1 : \frac{dW_t}{W_t} = (\delta\mu_P + (1-\delta)\mu_S)dt + \delta\sigma_P dZ_{P,t} + (1-\delta)\sigma_S dZ_{S,t} \tag{4.6}$$

$$t \in \tau_2 : \frac{dW_t}{W_t} = \mu_P dt + \sigma_P dZ_{P,t} \tag{4.7}$$

At first, we assume that not paid in capital in $t_0 = 0$ is invested fully in the risky asset and we further assume continuous rebalancing of the portfolio weights. This assumption is necessary in order to solve the problem analytically in closed form. If the whole commitment is paid in at the beginning, wealth follows the second process over the entire time

[32] In order to keep notations simple, the risky asset is assumed to be diluted or leveraged.

period. These processes allow implicitly continuous rebalancing as the fraction of private equity is constant over T. In this model context, this is only a weak assumption as will be shown later.

If an investor cannot meet the second capital call, he commits a default which is normally punished as described in section 2.2.3.

4.2 Modeling Opportunity Cost of Delayed Investment

In order to measure the loss of return due to the delayed investment flow, we calculate the certainty equivalent of expected final wealth denoted by p which equalizes expected wealth at the end of a Private Equity investment with and without delay. The effective final wealth from the Private Equity investment is denoted by W_T and the theoretical final wealth without delayed investment by $\bar{W_T}$. u(.) represents a CRRA-utility function (see equation 3.14):

$$E[u(W_T)] = E[u(e^{-pT}\bar{W_T})] \tag{4.8}$$

4.2.1 Opportunity Cost without Shortfall

To calculate p, we first need to solve for final wealth W_T using the differential equation of the log-wealth and Itô's Lemma. Without loss of generality we set $W_0 = 1$ and we assume that no shortfall occurs (or if so without consequences):

$$\begin{aligned} t \in \tau_1 : \mathrm{d}(lnW_t) &= \frac{dW_t}{W_t} - \frac{1}{2}(\frac{dW_t}{W_t})^2 \\ &= (\delta\mu_P - \frac{1}{2}\delta^2\sigma_P^2 + (1-\delta)\mu_S - \frac{1}{2}(1-\delta)^2\sigma_S^2 - \delta(1-\delta)\sigma_P\sigma_S\rho)dt \\ &+ \delta\sigma_P dZ_{P,t} + (1-\delta)\sigma_S dZ_{S,t} \end{aligned}$$

$$\Longrightarrow lnW_{t_1} = (\delta\mu_P - \frac{1}{2}\delta^2\sigma_P^2 + (1-\delta)\mu_S - \frac{1}{2}(1-\delta)^2\sigma_S^2 - \delta(1-\delta)\sigma_P\sigma_S\rho)t_1 + \delta\sigma_P Z_{P,t_1} + (1-\delta)\sigma_S Z_{S,t_1} \tag{4.9}$$

Again, the simplifying assumption is made that in t_1 the rest of the amount committed to the Private Equity fund can be invested. This leads to the dynamics in τ_2 with

$$t \in \tau_2 : \mathrm{d}(lnW_t) = (\mu_P - \frac{1}{2}\sigma_P^2)dt + \sigma_P dZ_{P,t}$$
$$\Longrightarrow lnW_T - lnW_{t_1} = (\mu_P - \frac{1}{2}\sigma_P^2)(T - t_1) + \sigma_P(Z_{P,T} - Z_{P,t_1}) \tag{4.10}$$

Inserting W_{t_1} yields

$$lnW_T = (\mu_P - \frac{1}{2}\sigma_P^2)(T - t_1) + (\delta\mu_P - \frac{1}{2}\delta^2\sigma_P^2 + (1-\delta)\mu_S - \frac{1}{2}(1-\delta)^2\sigma_S^2 - \delta(1-\delta)\sigma_P\sigma_S\rho)t_1 + \delta\sigma_P Z_{P,t_1} + (1-\delta)\sigma_S Z_{S,t_1} + \sigma_P(Z_{P,T} - Z_{P,t_1}) \tag{4.11}$$

For $\bar{W_T}$ results analogously to the dynamics in $t \in \tau_2$

$$ln\bar{W_T} = (\mu_P - \frac{1}{2}\sigma_P^2)T + \sigma_P Z_{P,T} \tag{4.12}$$

In order to ensure that an investor is willing to invest his entire endowments into the Private Equity fund, we have to calibrate γ assuming that the fraction x of total wealth invested in the private equity fund is equal to one. The Sharpe Ratio is denoted by SR. The importance of this calibration will be shown later:

$$x = \frac{\mu_P - r}{\gamma\sigma_P^2} = 1 \Longrightarrow \gamma = \frac{\mu_P - r}{\sigma_P^2} = \frac{SR_P}{\sigma_P} \tag{4.13}$$

In order to quantify the results throughout the following analysis, the necessary parameter values have to be specified. The parameters used are given in table 4.1[33]. These are valid throughout the whole analysis if not indicated else.

Parameter	Value
μ_P	0.20=20%
σ_P^2	0.12
μ_S	0.0925=9.25%
σ_S^2	0.0625
r	0.03=3%
γ	1.42
δ	0.30
ρ	0.5
α	0.75
T	10

Table 4.1: Model parameters

The Sharpe Ratios (SR) are $SR_P = 0.49$ for the Private Equity investment and $SR_S = 0.25$ for the public equity. We need the risk aversion to be $\gamma = \frac{SR_P}{\sigma_P}$ as defined in 4.13. Otherwise it is not rational for an investor to invest in the Private Equity investment according to his risk profile.

First, we calculate the expected utility of final wealth when the entire commitment is paid at the beginning of the investment horizon:

$$\begin{aligned} E[u(e^{-pT}\bar{W_T})] &= E[\frac{1}{1-\gamma}(e^{(\mu_P - p - \frac{1}{2}\sigma_P^2)T + \sigma_P Z_{P,T}})^{1-\gamma}] \\ &= \frac{1}{1-\gamma} e^{(1-\gamma)(\mu_P - p - \frac{1}{2}\sigma_P^2)T} E[e^{(1-\gamma)\sigma_P Z_{P,T}}] \\ &= \frac{1}{1-\gamma} e^{(1-\gamma)(\mu_P - p - \frac{1}{2}\sigma_P^2)T + \frac{1}{2}(1-\gamma)^2\sigma_P^2 T} \\ &= \frac{1}{1-\gamma} e^{(1-\gamma)(\mu_P - p - \frac{1}{2}\gamma\sigma_P^2)T} \end{aligned} \tag{4.14}$$

[33]The parameter α is introduced in later sections and refers to the penalty an investor has to pay in case of a default on commitment.

Formulating final wealth when the investment is split in two parts, we end up with[34]

$$E[u(W_T)] = \frac{1}{1-\gamma}\exp[(1-\gamma)(\mu_P - \frac{1}{2}\sigma_P^2)T + (1-\gamma)(\delta\mu_P - \mu_P - \frac{1}{2}\delta^2\sigma_P^2 + \frac{1}{2}\sigma_P^2 + (1-\delta)\mu_S - \frac{1}{2}(1-\delta)^2\sigma_S^2 - \delta(1-\delta)\sigma_P\sigma_S\rho)t_t] \\ E[\exp[(1-\gamma)(\delta\sigma_P Z_{P,t_1} + (1-\delta)\sigma_S Z_{S,t_1} + \sigma_P(Z_{P,T} - Z_{P,t_1}))]] \tag{4.15}$$

As Z_t has independent increments, only the correlation between Z_{P,t_1} and Z_{S,t_1} is different from zero and we have

$$E[u(W_T)] = \\ \frac{1}{1-\gamma}\exp[(1-\gamma)(\mu_P - \frac{1}{2}\sigma_P^2)T - (1-\gamma)(1-\delta)(\mu_P - \mu_S - \frac{1}{2}(1+\delta)\sigma_P^2 + \frac{1}{2}(1-\delta)\sigma_S^2 - \delta\sigma_P\sigma_S\rho)t_1 + \frac{1}{2}(1-\gamma)^2(\sigma_P^2(T-t_1) + \delta^2\sigma_P^2 t_1 + (1-\delta)^2\sigma_S^2 t_1 + 2\delta(1-\delta)\sigma_P\sigma_S\rho t_1)] \tag{4.16}$$

Equating the two expressions for final wealth in equations 4.14 and 4.16 and solving for p we get the following expression for the opportunity cost of delayed investment:

$$p = \frac{(1-\delta)t_1}{T}[\mu_P - \mu_S - \frac{1}{2}\gamma(1+\delta)\sigma_P^2 + \frac{1}{2}\gamma(1-\delta)\sigma_S^2 + \gamma\delta\sigma_P\sigma_S\rho] \tag{4.17}$$

The opportunity cost can be interpreted as the return loss p.a. which the investor incurs over the whole lifetime of the Private Equity investment. First, the condition on relative risk aversion from equation 4.13 is not yet taken into account. Some properties are easy to see: the earlier the second payment the lower the opportunity cost. The smaller the return

[34]For detailed derivations the reader is referred to Appendix C.3.

premium of the Private Equity investment and the higher its volatility the lower is the return loss. A higher correlation leads to higher opportunity cost as the diversification benefits are lower. For δ, the dynamics are not so obvious:

$$\frac{dp}{d\delta} = -\frac{t_1}{T}(\mu_P - \mu_S - \gamma\delta\sigma_P^2 + \gamma(1-\delta)\sigma_S^2 - \gamma(1-2\delta)\sigma_P\sigma_S\rho) \quad (4.18)$$

The opportunity cost are decreasing in δ when the following expression holds:

$$\frac{dp}{d\delta} < 0 \Longrightarrow \delta < \frac{\mu_P - \mu_S + \gamma\sigma_S^2 - \gamma\sigma_P\sigma_S\rho}{\gamma\sigma_P^2 + \gamma\sigma_S^2 - 2\gamma\sigma_P\sigma_S\rho} \quad (4.19)$$

As a result, for rather low levels of correlation there is a (rather high) level of δ for which opportunity cost are increasing for a larger share of investment in t_0. This is shown in table 4.2. For the correlation given in table 4.1, the critical value for δ is already extremely high. But the results show that in case of almost no correlation it can be beneficial for the investor if only around three quarter of the commitment is drawn at t_0.

ρ	Critical δ according to 4.19
0.5095	1.0000
0.50	0.9915
0.10	0.7853
0.00	0.7583

Table 4.2: Influence of δ on opportunity cost

The reason is that diversification benefits outweigh the larger expected returns from the Private Equity investment. In those cases opportunity cost can even be negative and the investment delay beneficial to overall return.

If we consider the several expressions for the opportunity cost p above, it is easy to see that $p < 0$ is also possible under certain conditions. To show this, we define the opportunity cost in case the whole gap is invested in the risk-less bond. The condition on γ from equation 4.13 is not used, yet:

$$p = \frac{(1-\delta)t_1}{T}[\mu_P - r - \frac{1}{2}\gamma(1+\delta)\sigma_P^2] \tag{4.20}$$

If the risk premium of the private equity investment does not reward for the associated risk weighted with the risk aversion coefficient γ and the magnitude of the second investment δ, then p can fall below 0. This can be the case if an investor is not willing to bear the risks of the private equity investment. As a result, a delayed investment path would be beneficial to him. This is shown in the following expressions:

$$\begin{aligned} \mu_P - r &< \frac{1}{2}\gamma(1+\delta)\sigma_P^2 \\ \gamma &> \frac{2(\mu_P - r)}{(1+\delta)\sigma_P^2} \end{aligned} \tag{4.21}$$

Therefore, if an investor's risk aversion γ does not satisfy 4.21, he is not willing to bear the risks of a Private Equity investment and the delayed investment path increases his utility which does not make sense in our optimization problem[35]. For the parameter values given in table 4.1, the relative risk aversion of an investor has to be smaller than $\gamma = 2.18$ in order to ensure that he has the risk profile and the ability to invest in a Private Equity fund. As this number is rather low, this study mainly focuses on institutional and large individual investors. As a result, we have to calibrate γ according to equation 4.13 in order to ensure that the investor is willing to invest his entire wealth in the Private Equity investment.

In the following step, we explicitly take the condition on γ from equation 4.13 into account. The expression for the opportunity cost from equation 4.17, in case the condition in 4.13 holds, changes to

[35] Under 4.13, the condition is never satisfied and the investor can always bear the risks of a Private Equity investment.

$$p = \frac{(1-\delta)t_1}{T}[\mu_P - \mu_S - \frac{1}{2}(\mu_P - r)(1+\delta) + \frac{1}{2}(\mu_P - r)(1-\delta)\frac{\sigma_S^2}{\sigma_P^2} + (\mu_P - r)\delta\frac{\sigma_S}{\sigma_P}\rho] \tag{4.22}$$

and analogously for equation 4.20 to

$$p = \frac{(\mu_P - r)(1-\delta)^2 t_1}{2T} \tag{4.23}$$

As the goal is to optimally allocate the investment gap in τ_1 between stocks and bonds, we introduce π as the fraction of capital invested in stocks. This changes the dynamics in τ_1 to

$$\frac{dW_t}{W_t} = (\delta\mu_P + (1-\delta)((1-\pi)r + \pi\mu_S))dt + \delta\sigma_P dZ_{P,t} + (1-\delta)\pi\sigma_S dZ_{S,t} \tag{4.24}$$

Opportunity cost then are[36]

$$p = \frac{(1-\delta)t_1}{T}[\mu_P - (1-\pi)r - \pi\mu_S - \frac{1}{2}\gamma(1+\delta)\sigma_P^2 + \frac{1}{2}\gamma(1-\delta)\pi^2\sigma_S^2 + \gamma\delta\sigma_P\pi\sigma_S\rho] \tag{4.25}$$

and analogously when condition 4.13 holds

$$p = \frac{(1-\delta)t_1}{T}[(r-\mu_S)\pi + (\mu_P - r)(1 - \frac{1}{2}(1+\delta) + \frac{1}{2}(1-\delta)\pi^2\frac{\sigma_S^2}{\sigma_P^2} + \delta\pi\frac{\sigma_S}{\sigma_P}\rho)] \tag{4.26}$$

Minimizing opportunity cost (without shortfall, NSF) with respect to π leads to

$$\pi^*_{NSF} = \frac{\mu_S - r - \gamma\delta\sigma_S\sigma_P\rho}{\gamma(1-\delta)\sigma_S^2} = \frac{\mu_S - r}{\gamma(1-\delta)\sigma_S^2} - \frac{\delta\sigma_P\rho}{(1-\delta)\sigma_S} \tag{4.27}$$

[36]The result can be seen easily if we make the following changes in equation 4.17: $\mu_S \to (1-\pi)r + \pi\mu_S$ and $\sigma_S \to \pi\sigma_S$.

Under condition 4.13 on γ one obtains

$$\pi^*_{NSF} = \frac{(\mu_S - r)\sigma_P^2}{(\mu_P - r)(1-\delta)\sigma_S^2} - \frac{\delta\sigma_P\rho}{(1-\delta)\sigma_S} \tag{4.28}$$

The first term in equation 4.27 is similar to the fraction of risky assets in the Merton portfolio enlarged by $(1-\delta)$ in the nominator. The fraction of the risky investment is reduced by the correlation of public and private equity returns. It is an interesting question to examine when the fraction of risky investments in this setting is larger than in the Merton portfolio $(\gamma > 1)$. We end up with the following condition:

$$\pi^*_{NSF} > \pi_M \Longrightarrow \rho < \frac{\mu_S - r}{\gamma\sigma_P\sigma_S} \overset{\text{Using (4.13)}}{\Longrightarrow} \rho < \frac{\mu_S - r}{\sigma_S}\frac{\sigma_P}{\mu_P - r} = \frac{SR_S}{SR_P} \tag{4.29}$$

It is interesting to see that ρ has to be smaller than the SR-ratio for $\pi^*_{NSF} > \pi_M$ to hold. For the parameter values given the result is $\rho < 0.5095$. Only for higher levels of correlation the fraction in public equity is smaller in this model setting than for the Merton portfolio. As a result, we can conclude that investing in the Merton portfolio bears too many risks in case of rather high correlation and for lower correlation a more risky strategy is strictly better. We will show that this result remains robust when including a positive probability of a default on commitment. Also interesting is the fact that π^* is constant over time and independent of the timing of the second payment.

Figure 4.1 shows the level of opportunity cost for $\pi = 0, 1$ and π^*.

For the parameter values given, π^*_{NSF} is equal to 71.15%. This is only slightly above the Merton portfolio weight with $\pi_M = 70.59\%$. Table 4.3 shows several values of π^*_{NSF} depending on the coefficient of correlation ρ. For the parameter values given, 4.29 holds for $\rho < 50.94\%$.

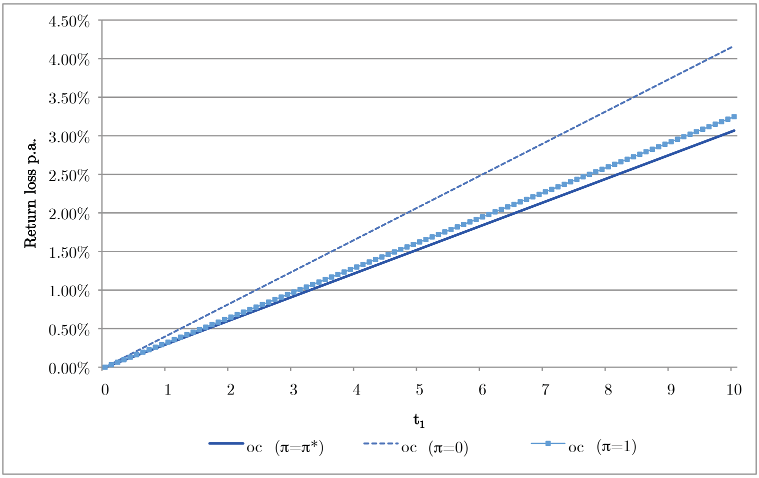


Figure 4.1: Opportunity cost

ρ	Optimal weight of risky assets
1.0	0.4146
0.9	0.4739
0.7	0.5927
0.5	0.7115
0.1	0.9490
0.0	1.0084

Table 4.3: Optimal weights of risky assets without shortfall

4.2.2 Shortfall Probability of a Default on Commitment

A default on commitment occurs if an investor has not enough funds to cover the second capital call. This can occur if the investment gap from late investment is (partially) invested in risky assets. If we define an investment of the gap entirely in the risk-less asset as the benchmark case, a default occurs if the return of the gap is smaller than the risk-free rate. As the goal is to optimize the overall investment, the risk-less return opportunity has to be taken into consideration and therefore the time-value of the second call of commitment. It is obvious that this condition increases the shortfall probability anything else being equal. This leads to the expression

$$(1-\delta)C_0 e^{((1-\pi)r+\pi\mu_S-\frac{1}{2}\pi^2\sigma_S^2)t_1+\pi\sigma_S Z_{S,t_1}} < (1-\delta)C_0 e^{rt_1} \tag{4.30}$$

Solving equation 4.30 for Z_{S,t_1} leads to a standard formulation of the shortfall probability. As this transformation includes the division by π, we need $\pi > 0$ to hold for the inequality to remain valid. As a result, shorting the risky asset (when investing the gap) is not allowed in this model. Therefore, the shortfall probability P_{SF} can be described as

$$\begin{aligned} P_{SF,BM}[Z_{S,t_1} &< \frac{(\pi r - \pi\mu_S + \frac{1}{2}\pi^2\sigma_S^2)t_1}{\pi\sigma_S}] \\ &= \Phi[\frac{(\pi r - \pi\mu_S + \frac{1}{2}\pi^2\sigma_S^2)t_1}{\pi\sigma_S\sqrt{t_1}}] = \Phi[\underbrace{\frac{r-\mu_S}{\sigma_S}\sqrt{t_1} + \frac{1}{2}\pi\sigma_S\sqrt{t_1}}_{z}] \end{aligned} \tag{4.31}$$

where the index BM indicates the consideration of the risk-free rate when defining the shortfall. Without taking the time-value of the second commitment call into account and therefore not considering the risk-less return as a benchmark, a shortfall only occurs if the gap in t_1 is smaller than $(1-\delta)C_0$ and the definition of the shortfall probability in 4.31 obviously changes to

$$P_{SF,NBM} = \Phi[\frac{r-\mu_S}{\sigma_S}\sqrt{t_1} - \frac{r}{\pi\sigma_S}\sqrt{t_1} + \frac{1}{2}\pi\sigma_S\sqrt{t_1}] \tag{4.32}$$

This distinction in defining a shortfall has very interesting implications for the analysis. From 4.32 it can be seen easily that z is always smaller when considering the time-value which can also be seen from figure 4.3. It will be shown later that the opportunity cost depend directly on P_{SF}. The expression for the shortfall probability in 4.32 is continuous in π (for $\pi \geq 0$) and therefore reducing the risky asset can reduce the shortfall probability continuously and minimizing the opportunity cost guarantees a single optimal point.

Furthermore, it can be shown easily that for realistic parameter values 4.31 and 4.32 are always below 50%. We assume here that an investor does not invest in a risky asset if its expected log-return is less than the risk-free rate.

Proof:

$P_{SF} < 0.5$ holds if $z < 0$. This leads to the following condition for $P_{SF,BM}$

$$\begin{aligned} &\frac{r-\mu_S}{\sigma_S}\sqrt{t_1} + \frac{1}{2}\pi\sigma_S\sqrt{t_1} < 0 \\ &\mu_S - \frac{1}{2}\pi\sigma_S^2 > r \end{aligned} \tag{4.33}$$

The expected (log-)return of the risky asset (with the variance weighted by π) has to be larger than the risk-free rate. For realistic parameter values this is always true. Using the parameter values of table 4.1, the condition is $\pi < 2$. The result is that $P_{SF,BM}$ is not only continuous in π but due to the above proof it is shown to be continuously increasing in π if $\pi < 2$ holds. This has interesting implications for the analysis of the opportunity cost in later sections. It is straightforward to show that $P_{SF,NBM} < 0.5$ holds analogously, reducing the right hand side of 4.33 by $(1 - \frac{1}{\pi})$.

In order to get an understanding on the magnitude of the shortfall risk, it is displayed graphically. Figure 4.2 shows the shortfall probability depending on t_1 with and without considering the risk-less investment as a benchmark case for the optimal investment path π^* derived above. In Figure 4.3 time t_1 equals 5 and the shortfall probability depends on $0 \leq \pi \leq 1$. It is intuitive that both expressions of P_{SF} increase with a more risky investment strategy. The consideration of the risk-less benchmark is substantial when evaluating the shortfall probability as can be seen from both graphs. The benchmark also causes $P_{SF,BM}$ to be larger than zero for $t_1 = 0$.

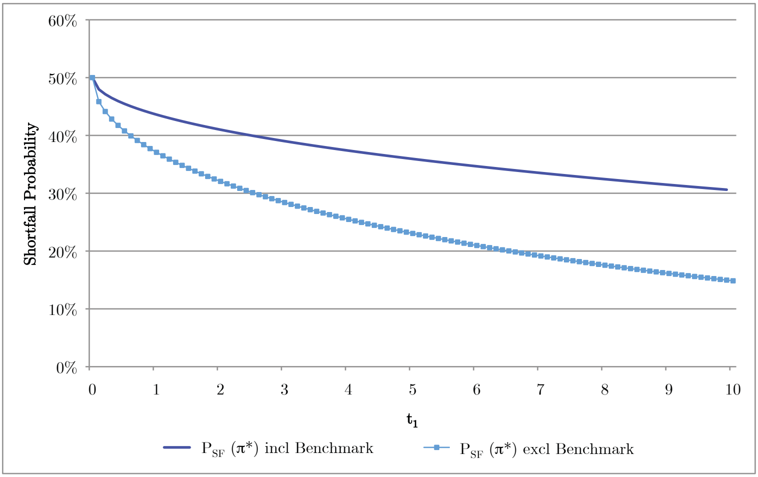


Figure 4.2: Shortfall probability with fixed $\pi = \pi^*$

The overall probability of default seems to be quite high especially including the risk-less return into the definition of a default. But we have to keep in mind that risk aversion is rather low using the parameters of this model ($\gamma = 1.42$) and we have an investor willing to invest all his wealth into Private Equity accepting the corresponding risks. At first, it might be surprising that the curves are decreasing in t_1. But when the second payment into the Private Equity fund comes very soon, early drawbacks on the risky investment are weighted extremely high as they are followed by only a very short recovery period. Therefore, we have to weigh α and the dynamics in the time interval τ_2 with the unknown shortfall probability P_{SF} in order to get an understanding of its implications on optimal investments which will be done in section 5.4.

4.2.3 Opportunity Cost with Certain Default

The consequences of a default on commitment are outlined in a fund's organizational documents. Typical effects are that an investor is excluded from the fund and the funds already invested are refunded applying a

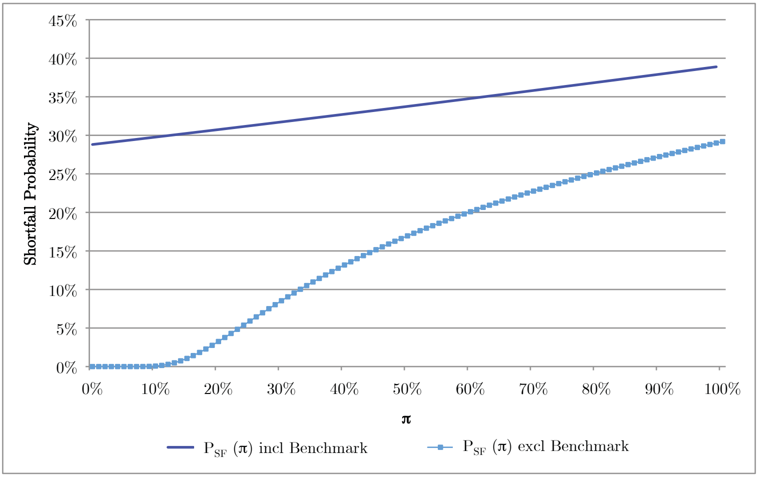


Figure 4.3: Shortfall probability for $t_1 = 5$, $0 \leq \pi \leq 1$

discount α[37]. In this model setting a shortfall has therefore two consequences. First, in t_1 the investor is only left with a fraction α of the money invested in the Private Equity fund. The rest is taken away as an administration and penalty fee. Second, in $t \in \tau_2$ the investor can no more profit from the returns of the Private Equity fund. As a result, he can only invest in public equity and bonds. Therefore, it is assumed that the optimal strategy in $t \in \tau_2$ is to invest in the Merton portfolio. If we assume that a default on commitment occurs for sure, it is equivalent to a reduction of initial wealth as the investor has to pay the penalty with certainty. As a result, initial wealth ($W_0 = C_0$) now changes to

$$\tilde{W}_0 = C_0 - (1-\alpha)\delta C_0 = \alpha\delta C_0 + (1-\delta)C_0 = (1-(1-\alpha)\delta)C_0 \quad (4.34)$$

as the investor has to pay a penalty and can only recover the fraction α from the funds invested in Private Equity and the rest, $(1-\alpha)\delta C_0$, is lost. The weight of assets that are invested in Private Equity at $t_0 = 0$ therefore also changes to

[37] See section 2.2.3.

$$\frac{\alpha\delta C_0}{\tilde{W_0}} = \frac{\alpha\delta}{1-(1-\alpha)\delta} = \bar{\delta} \tag{4.35}$$

with the remainder, $(1-\bar{\delta})$, being invested in stocks and bonds. Similar to the derivations above, we assume at first that the remainder is invested entirely in stocks. This leads to the following wealth processes for the two time periods:

$$t \in \tau_1 : \frac{dW_t}{W_t} = (\bar{\delta}\mu_P + (1-\bar{\delta})\mu_S)dt + \bar{\delta}\sigma_P dZ_{P,t} + (1-\bar{\delta})\sigma_S dZ_{S,t} \tag{4.36}$$

$$t \in \tau_2 : \frac{dW_t}{W_t} = ((1-\pi_M)r + \pi_M\mu_S)dt + \pi_M\sigma_S dZ_{S,t} \tag{4.37}$$

As $W_0 = C_0$ is again set to one, we have in reduced form $ln\tilde{W_0} = ln(1-(1-\alpha)\delta)$. The calculations leading to final log-wealth are exactly the same than in section 4.2.1 and yield:

$$\begin{aligned} lnW_T = {} & ((1-\pi_M)r + \pi_M\mu_S - \frac{1}{2}\pi_M^2\sigma_S^2)(T-t_1) + (\bar{\delta}\mu_P - \frac{1}{2}\bar{\delta}^2\sigma_P^2 \\ & + (1-\bar{\delta})\mu_S - \frac{1}{2}(1-\bar{\delta})^2\sigma_S^2 - \bar{\delta}(1-\bar{\delta})\sigma_P\sigma_S\rho)t_1 + ln(1-(1-\alpha)\delta) \\ & + \bar{\delta}\sigma_P Z_{P,t_1} + (1-\bar{\delta})\sigma_S Z_{S,t_1} + \pi_M\sigma_S(Z_{S,T} - Z_{S,t_1}) \end{aligned} \tag{4.38}$$

This leads to the following expression for the opportunity cost in case a default occurs with $P_{SF} = 1$[38]:

$$\begin{aligned} p = {} & \mu_P - \pi_M\mu_S - (1-\pi_M)r - \frac{1}{2}\gamma(\sigma_P^2 - \pi_M^2\sigma_S^2) + \frac{ln(1-(1-\alpha)\delta)}{T} \\ & + (\pi_M\mu_S + (1-\pi_M)r - \frac{1}{2}\gamma\pi_M^2\sigma_S^2 - \bar{\delta}\mu_P + \frac{1}{2}\bar{\delta}^2\gamma\sigma_P^2 - (1-\bar{\delta})\mu_S \\ & + \frac{1}{2}(1-\bar{\delta})^2\gamma\sigma_S^2 + \bar{\delta}(1-\bar{\delta})\gamma\sigma_P\sigma_S\rho)\frac{t_1}{T} \end{aligned} \tag{4.39}$$

[38] For detailed derivations the reader is referred to Appendix C.4.

Minimizing the resulting opportunity cost in case of a shortfall with respect to π yields

$$\pi^*_{SF} = \frac{\mu_S - r - \gamma\bar{\delta}\sigma_S\sigma_P\rho}{\gamma(1-\bar{\delta})\sigma_S^2} \overset{\text{Using (4.35)}}{=} \frac{(\mu_S - r)(1-(1-\alpha)\delta) - \gamma\alpha\delta\sigma_S\sigma_P\rho}{\gamma(1-\delta)\sigma_S^2} \tag{4.40}$$

The optimal result is similar to the case without shortfall but, as expected, does now depend on the penalty in case of a default, α. It does not depend on the changing dynamics in τ_2. This finding might be quite surprising at first sight. But if we look at the expression for the opportunity cost, it becomes clear that the inferior investment opportunities after a shortfall (resulting in the changing dynamics in τ_2) shift the level of opportunity cost but cannot be influenced by the investor's decision of π^*. The only thing an investor can control is the event risk. But if an investor changes π^*, he changes the expected shortfall probability and also (in the opposite direction) the return expectation. The simulations in chapter 5.4 will show the influence of the shortfall probability on the optimal investment path.

If the optimal solution is compared to the solution without shortfall (see 4.27), the following condition holds:

$$\pi^*_{SF} > \pi^*_{NSF} \Longrightarrow \rho > \frac{\mu_S - r}{\gamma\sigma_S\sigma_P} \overset{\text{Using (4.13)}}{=} \frac{SR_S}{SR_P} \tag{4.41}$$

As a result, for rather high levels of correlation, the optimal investment path is more risky in case a shortfall occurs with certainty. The reason is that the higher the correlation, the lower are diversification benefits and the lower is the effect of the changing dynamics in τ_2, where the investor has no more Private Equity exposure in case of a default. The same condition on ρ holds for $\pi^*_{SF} < \pi_M$. If the correlation is higher than the SR-ratio, investing in the Merton portfolio bears more risk than the optimal solution. This condition is equivalent to the case without shortfall (see 4.29).

Figure 4.4 shows the sensitivity of π^*_{SF} to changes in α, the cost of a default on commitment, for different levels of correlation. As $\rho = 0.5$ is close to the critical value from condition 4.41, the optimal investment path does practically not react to changes in α. For low levels of correlation, the optimal solution becomes riskier if the cost of a default decrease and vice versa (the lower α, the higher the cost of a default).

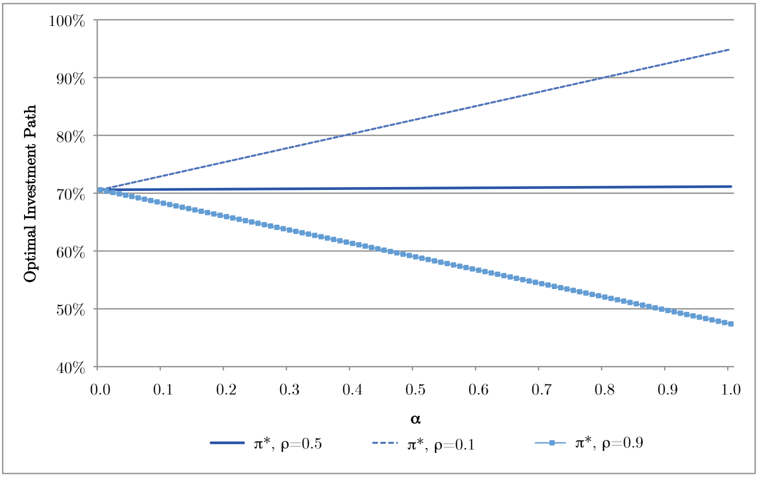


Figure 4.4: Optimal investment path, $P_{SF} = 1$

We have seen above that the opportunity cost of late investment evolve according to equations 4.25 for $P_{SF} = 0$ and 4.39 for $P_{SF} = 1$. The crucial assumption to solve the problem analytically was continuous rebalancing. As the three described asset classes have much different expected returns and risks, it can be supposed that the calculated costs will differ more from the true value the later the second investment in the Private Equity fund will take place. As Private Equity has the highest expected return, its impact on final wealth will be understated assuming continuous rebalancing and the analytical solution is supposed to overstate opportunity cost. This overstatement will be the lower the smaller the difference of the relative risk premia of the three investments is. On the other hand, continuous rebalancing is not an unrealistic assumption. Most Private

Equity investors are large institutional investors or high net-worth individuals. Both have strict investment guidelines either as legal obligations or as internally developed optimal investment strategies also fixing asset weights. As a result, the investor is always trying to meet his requirements keeping weights constant. Due to the illiquidity of Private Equity investments and the property of late investment this can never be reached exactly but evidence shows that professional asset managers come very close to it, e.g. using rolling investment strategies and publicly traded Private Equity investment funds. Having this in mind, continuous rebalancing is, at least for large investors, a quite realistic assumption[39].

4.3 Multiple Investment Rounds

During the analysis above, the assumption was made that the commitment is invested in two separate investment rounds. The first part at the beginning (δ) and the remainder at time t_1. As a result, the analysis was split in two time periods. Relaxing this assumption and assuming that the commitment is paid in within three investment rounds (at t_0, t_1 and t_2), leads to three time periods with three different wealth processes. Then, in t_2 all capital is paid in. In the following lines, the model is extended to three time periods with three separate investment rounds into the Private Equity fund. The solution is again derived analytically. The extension to n investment rounds is then straightforward. Analogously to equations 4.6 and 4.7 the three wealth processes are

$$t \in \tau_1 = [0, t_1] : \frac{dW_t}{W_t} = (\delta_1 \mu_P + (1-\delta_1)\mu_S)dt + \delta_1 \sigma_P dZ_{P,t} + (1-\delta_1)\sigma_S dZ_{S,t} \tag{4.42}$$

$$\begin{aligned} t \in \tau_2 = [t_1, t_2] : \frac{dW_t}{W_t} &= ((\delta_1 + \delta_2)\mu_P + (1 - \delta_1 - \delta_2)\mu_S)dt \\ &+ (\delta_1 + \delta_2)\sigma_P dZ_{P,t} + (1 - \delta_1 - \delta_2)\sigma_S dZ_{S,t} \end{aligned} \tag{4.43}$$

[39] See section 2.2.3.

$$t \in \tau_3 = [t_2, T] : \frac{dW_t}{W_t} = \mu_P dt + \sigma_P dZ_{P,t} \tag{4.44}$$

Committed but not yet invested funds are again assumed to be invested completely in the risky asset. From the wealth processes the dynamics can be derived and we have

$$lnW_T - lnW_{t_2} = (\mu_P - \frac{1}{2}\sigma_P^2)(T - t_2) + \sigma_P(Z_{P,T} - Z_{P,t_2}) \tag{4.45}$$

$$\begin{aligned} lnW_{t_2} - lnW_{t_1} = {} & ((\delta_1 + \delta_2)\mu_P - \frac{1}{2}(\delta_1 + \delta_2)^2\sigma_P^2 + (1 - \delta_1 - \delta_2)\mu_S \\ & - \frac{1}{2}(1 - \delta_1 - \delta_2)^2\sigma_S^2 - (\delta_1 + \delta_2)(1 - \delta_1 - \delta_2)\sigma_P\sigma_S\rho)(t_2 - t_1) \\ & + (\delta_1 + \delta_2)\sigma_P(Z_{P,t_2} - Z_{P,t_1}) + (1 - \delta_1 - \delta_2)\sigma_S(Z_{S,t_2} - Z_{S,t_1}) \end{aligned} \tag{4.46}$$

$$\begin{aligned} lnW_{t_1} = {} & \\ & (\delta_1\mu_P - \frac{1}{2}\delta_1^2\sigma_P^2 + (1 - \delta_1)\mu_S - \frac{1}{2}(1 - \delta_1)^2\sigma_S^2 - \delta_1(1 - \delta_1)\sigma_P\sigma_S\rho)t_1 \\ & + \delta_1\sigma_P Z_{P,t_1} + (1 - \delta_1)\sigma_S Z_{S,t_1} \end{aligned} \tag{4.47}$$

again setting $W_0 = 1$. The fractions of committed capital invested during the three rounds are δ_1, δ_2 and $(1 - \delta_1 - \delta_2)$. The total amount of committed capital, C_0, is again set equal to one. Summing up equations 4.45 to 4.47, lnW_T can be calculated. The shortfall probability is again assumed to be zero. This leads to the expected utility of final wealth[40]

[40]The calculations are very similar to the ones in section 4.2.1 and Appendix C.3.

$$E[u(W_T)] = \frac{1}{1-\gamma}\exp[(1-\gamma)((\mu_P - \frac{1}{2}\gamma\sigma_P^2)(T-t_2) + ((\delta_1+\delta_2)\mu_P$$
$$- \frac{1}{2}\gamma(\delta_1+\delta_2)^2\sigma_P^2 + (1-\delta_1-\delta_2)\mu_S - \frac{1}{2}\gamma(1-\delta_1-\delta_2)^2\sigma_S^2$$
$$- \gamma(\delta_1+\delta_2)(1-\delta_1-\delta_2)\sigma_P\sigma_S\rho)(t_2-t_1) + (\delta_1\mu_P - \frac{1}{2}\gamma\delta_1^2\sigma_P^2$$
$$+ (1-\delta_1)\mu_S - \frac{1}{2}\gamma(1-\delta_1)^2\sigma_S^2 - \gamma\delta_1(1-\delta_1)\sigma_P\sigma_S\rho)t_1)] \tag{4.48}$$

The expected utility of final wealth, assuming the entire commitment is paid in at $t_0 = 0$, is equal to equation 4.14:

$$E[u(e^{-pT}\bar{W}_T)] = \frac{1}{1-\gamma}e^{(1-\gamma)(\mu_P - p - \frac{1}{2}\gamma\sigma_P^2)T}$$

Equating the expressions for the utility of final wealth (4.14 and 4.48) and assuming that not yet paid in commitments are invested in stocks and bonds (by changing μ_S to $\pi\mu_S + (1-\pi)r$ and σ_S to $\pi\sigma_S$ in order to introduce π), leads to an expression for the opportunity cost p in case the commitment is paid in to the Private Equity fund within three investment rounds[41]:

$$p = (\mu_P - \frac{1}{2}\gamma\sigma_P^2)\frac{t_2}{T} - ((\delta_1+\delta_2)\mu_P - \frac{1}{2}\gamma(\delta_1+\delta_2)^2\sigma_P^2$$
$$+ (1-\delta_1-\delta_2)(\pi\mu_S + (1-\pi)r) - \frac{1}{2}\gamma(1-\delta_1-\delta_2)^2\pi^2\sigma_S^2$$
$$- \gamma(\delta_1+\delta_2)(1-\delta_1-\delta_2)\sigma_P\pi\sigma_S\rho)\frac{(t_2-t_1)}{T} - (\delta_1\mu_P - \frac{1}{2}\gamma\delta_1^2\sigma_P^2$$
$$+ (1-\delta_1)(\pi\mu_S + (1-\pi)r) - \frac{1}{2}\gamma(1-\delta_1)^2\pi^2\sigma_S^2 - \gamma\delta_1(1-\delta_1)\sigma_P\pi\sigma_S\rho)\frac{t_1}{T} \tag{4.49}$$

Minimizing the opportunity cost with respect to π leads to

[41] The reader is referred to Appendix C.3 for similar calculations.

$$\pi^* = \frac{((1-\delta_1-\delta_2)(\mu_S - r) - \gamma(\delta_1+\delta_2)(1-\delta_1-\delta_2)\sigma_P\sigma_S\rho)(t_2-t_1)}{\gamma\sigma_S^2((1-\delta_1-\delta_2)^2(t_2-t_1)+(1-\delta_1)^2 t_1)}$$

$$+\frac{((1-\delta_1)(\mu_S - r) - \gamma\delta_1(1-\delta_1)\sigma_P\sigma_S\rho)t_1}{\gamma\sigma_S^2((1-\delta_1-\delta_2)^2(t_2-t_1)+(1-\delta_1)^2 t_1)} \tag{4.50}$$

The expression is similar to the case with two investments into the fund and the sensitivities of $\gamma, \sigma_S, \sigma_P, \mu_S, r$ and ρ have the same sign as above. It is obvious from equation 4.49 that opportunity cost, and therefore the resulting optimal investment path, depend on the time schedule of the three payments and also on the amount that is paid in each time. Therefore, the optimal investment path π^* depends on the time periods between the payments and the corresponding investment amounts. The second part of the fraction in equation 4.50 is similar to the solution derived above only adding the term for the additional investment round in the nominator. As a result, an investor needs to know not only the timing of the several investments but also their amount in order to optimize his investment behavior. As was shown in several studies, it is possible to estimate the potential investment path of a fund from single fund data with adequate precision[42]. The results are similar to the pattern in figure 2.8. If we assume that the time intervals are of equal length, it is obvious that the respective expressions cancel out and the optimal investment path only depends on the size of the singular investments but not on the timing. The assumption only needs the time interval between the payments to be equal, the length of the interval does not matter. Furthermore, the above formula can also be derived for n investment rounds into the fund in a straightforward way. As a result, if an investor can consistently estimate the capital calls of a Private Equity investment, the above formula gives a closed-form solution for the optimal investment of the committed but not yet invested funds. This result also holds if Private Equity is only

[42] See section 2.2.3 and especially Frei & Studer (2003).

a smaller part in an overall portfolio and the shortfall probability tends towards zero as will be shown in later sections.

5 Private Equity in a Diversified Portfolio

It is a logical consequence of the inclusion of Private Equity investments in a mixed portfolio consisting of Private Equity, public equity and bonds that the effect of a shortfall plays only a minor role depending on the weight of Private Equity. The question is in which cases and under what conditions the shortfall probability will decrease to zero. Even if the shortfall probability is zero, it is nevertheless important to optimally allocate the investment gap from the Private Equity investment on stocks and bonds and optimize the positions of the conventional investments. In this section, we will extent the model, evaluate the influence of a shortfall and simulate the optimal weights for stocks and bonds when investing the gap within a mixed portfolio context.

5.1 Model Extensions

Let the initial portfolio investments at time $t_0 = 0$ in Private Equity, public equity and bonds be C_0, S_0 and B_0 respectively summing up to initial wealth W_0. Of course, C_0 is the whole initial commitment and not the actual investment. The corresponding portfolio weights are straightforward

$$x = \frac{C_0}{C_0 + S_0 + B_0} \tag{5.1}$$

$$s = \frac{S_0}{C_0 + S_0 + B_0} \tag{5.2}$$

$$b = \frac{B_0}{C_0 + S_0 + B_0} \tag{5.3}$$

summing up to one. In a first approach, we assume that the equity and bond parts of the portfolio $(s+b)$ are invested according to Merton's optimization rule (see equation 3.12). Therefore s and b are also equal to

$$s = (1-x)\pi_M$$

$$b = (1-x)(1-\pi_M)$$

As for shortfall considerations only the liquid parts of the overall portfolio matter, we have to exclude the investment in the Private Equity fund at time $t_0 = 0$, δC_0. Therefore, liquid wealth $\bar{W}_0$ is $(1-\delta)C_0 + S_0 + B_0$ and the above weights change to

$$\bar{x} = \frac{(1-\delta)C_0}{(1-\delta)C_0 + S_0 + B_0} \tag{5.4}$$

$$\bar{s} = \frac{S_0}{(1-\delta)C_0 + S_0 + B_0} \tag{5.5}$$

$$\bar{b} = \frac{B_0}{(1-\delta)C_0 + S_0 + B_0} \tag{5.6}$$

again summing up to one. We can relate the two respective expressions dividing by W_0:

$$\bar{x} = \frac{(1-\delta)x}{(1-\delta)x + s + b} = \frac{(1-\delta)x}{1-\delta x} \tag{5.7}$$

$$\bar{s} = \frac{s}{(1-\delta)x + s + b} = \frac{(1-x)\pi_M}{1-\delta x} \tag{5.8}$$

$$\bar{b} = \frac{b}{(1-\delta)x + s + b} = \frac{(1-x)(1-\pi_M)}{1-\delta x} \tag{5.9}$$

Setting $(1-\delta)C_0 = \bar{C}_0$ and defining its dynamics we get

$$\frac{d\bar{C}_t}{\bar{C}_t} = (\pi\mu_S + (1-\pi)r)dt + \pi\sigma_S dZ_{S,t} \tag{5.10}$$

with π being the amount of the investment gap invested in the risky asset. The dynamics for S and B remain the same as in section 4.1.

Again, we simplify the analysis in the same way than above assuming constant rebalancing resulting in constant portfolio weights over time. If we consider the portfolio of a large institutional investor which is normally bounded by certain investment guidelines defining the target weights of the different asset classes, this assumption is not far from reality at least in the long-run.

5.2 Shortfall Probability in a Mixed Portfolio Context

If an investor is not only invested in Private Equity but also in public equity and risk-free bonds, a default on commitment only occurs if liquid wealth at time t_1 is less than the second part of the private equity investment $(1-\delta)C_0 = \bar{C_0}$. We assume that there are no liquidation costs of public equity and risk-free bonds. As we still consider the risk-free invested gap as the benchmark case, a shortfall occurs if liquid wealth W_{t_1} at time t_1 (when the second part of the commitment is called) is lower than the required amount of the second commitment invested at the risk-free rate r. The risk-free return is again considered as the benchmark case:

$$\bar{C_{t_1}} + S_{t_1} + B_{t_1} < \bar{C_0} e^{rt_1} \tag{5.11}$$

Dividing by $\bar{W_0}$ and taking logs leads to

$$\begin{aligned} &\frac{\bar{C_{t_1}} + S_{t_1} + B_{t_1}}{\bar{W_0}} < \bar{x} e^{rt_1} \\ &ln(\frac{\bar{C_{t_1}} + S_{t_1} + B_{t_1}}{\bar{W_0}}) = ln(W_{t_1}) < ln\bar{x} + rt_1 \end{aligned} \tag{5.12}$$

Using the dynamics of $\bar{C_t}$, S_t and B_t we can calculate the dynamics of d$ln(W_t)$ using Itô's Lemma:

$$
\begin{aligned}
&\mathrm{d}ln(W_t) = \\
&\frac{\bar{W_0}}{\bar{C_t}+S_t+B_t}\frac{1}{\bar{W_0}}d\bar{C_t} + \frac{\bar{W_0}}{\bar{C_t}+S_t+B_t}\frac{1}{\bar{W_0}}dB_t + \frac{\bar{W_0}}{\bar{C_t}+S_t+B_t}\frac{1}{\bar{W_0}}dS_t \\
&-\frac{1}{2}\frac{1}{(\bar{C_t}+S_t+B_t)^2}(d\bar{C_t})^2 - \frac{1}{2}\frac{1}{(\bar{C_t}+S_t+B_t)^2}(dB_t)^2 \\
&-\frac{1}{2}\frac{1}{(\bar{C_t}+S_t+B_t)^2}(dS_t)^2 - \frac{1}{2}\frac{1}{(\bar{C_t}+S_t+B_t)^2}2d\bar{C_t}dS_t
\end{aligned}
\tag{5.13}
$$

Terms of higher order in dt are set equal to 0. As dB_t does not depend on a stochastic part, $(dB_t)^2$ can also be set to 0. After canceling out and replacing the fractions with the corresponding portfolio weights of the liquid portfolio, which are assumed to be constant, we get[43]

$$
\begin{aligned}
\mathrm{d}ln(W_t) = &\bar{x}\frac{d\bar{C_t}}{\bar{C_t}} + \bar{s}\frac{dS_t}{S_t} + \bar{b}\frac{dB_t}{B_t} - \frac{1}{2}\bar{x}^2(\frac{d\bar{C_t}}{\bar{C_t}})^2 - \frac{1}{2}\bar{s}^2(\frac{dS_t}{S_t})^2 - \frac{1}{2}\bar{b}^2(\frac{dB_t}{B_t})^2 \\
&- \bar{x}\bar{s}\frac{d\bar{C_t}}{\bar{C_t}}\frac{dS_t}{S_t}
\end{aligned}
\tag{5.14}
$$

Replacing the dynamics with the corresponding differential equations leads to

$$
\begin{aligned}
\mathrm{d}ln(W_t) = &[(1-\pi)\bar{x}r + \pi\bar{x}\mu_S - \frac{1}{2}\bar{x}^2\pi^2\sigma_S^2 + \bar{s}\mu_S - \frac{1}{2}\bar{s}^2\sigma_S^2 + \bar{b}r - \bar{x}\bar{s}\pi\sigma_S^2]dt \\
&+ (\pi\bar{x} + \bar{s})\sigma_S dZ_{S,t}
\end{aligned}
\tag{5.15}
$$

Using these dynamics, the condition for a shortfall becomes

$$
ln(W_{t_1}) - rt_1 < ln\bar{x}
$$

[43]For detailed derivations the reader is referred to Appendix C.5.

$$[(1-\pi)\bar{x}r + \pi\bar{x}\mu_S - \frac{1}{2}\bar{x}^2\pi^2\sigma_S^2 - r + \bar{s}\mu_S - \frac{1}{2}\bar{s}^2\sigma_S^2 + \bar{b}r - \bar{x}\bar{s}\pi\sigma_S^2]t_1 + (\pi\bar{x}+\bar{s})\sigma_S Z_{S,t_1} < ln\bar{x} \tag{5.16}$$

Rearranging and simplifying leads to

$$[(\mu_S - r)(\pi\bar{x}+\bar{s}) - \frac{1}{2}\sigma_S^2(\pi\bar{x}+\bar{s})^2]t_1 + (\pi\bar{x}+\bar{s})\sigma_S Z_{S,t_1} < ln\bar{x} \tag{5.17}$$

Solving for the stochastic component $Z_{S,t} \sim N(0,\sqrt{t})$ leads to the input argument z for the standardnormal-cdf Φ on the right hand side:

$$\frac{Z_{S,t_1}}{\sqrt{t_1}} < \frac{ln\bar{x}}{(\pi\bar{x}+\bar{s})\sigma_S\sqrt{t_1}} - \frac{[\mu_S - r - \frac{1}{2}\sigma_S^2(\pi\bar{x}+\bar{s})]t_1}{\sigma_S\sqrt{t_1}} = z \tag{5.18}$$

The shortfall probability equals

$$P_{SF,BM}(\bar{C_{t_1}} + S_{t_1} + B_{t_1} < \bar{C_0}e^{rt_1}) = \Phi(z) \tag{5.19}$$

Without taking the benchmark r into account, z changes only slightly, removing the term $-r$ from the numerator in 5.18.

5.3 Evaluation of the Shortfall Probability

If we calculate the shortfall probability in a mixed portfolio context using the common parameter values, we come to the not very surprising conclusion that a default on commitment becomes extremely rare for the usual portfolio weights of Private Equity of around 10%-20%. The following graphs show the shortfall probability with and without the risk-free benchmark dependent on the time lag of the second payment for five different Private Equity weights in the overall portfolio. The remaining funds are assumed to be invested in public equity and risk-free bonds. An optimization model for the overall portfolio will be derived in section 5.6.

The investment gap is assumed to be invested according to π^* derived above (equation 4.40).

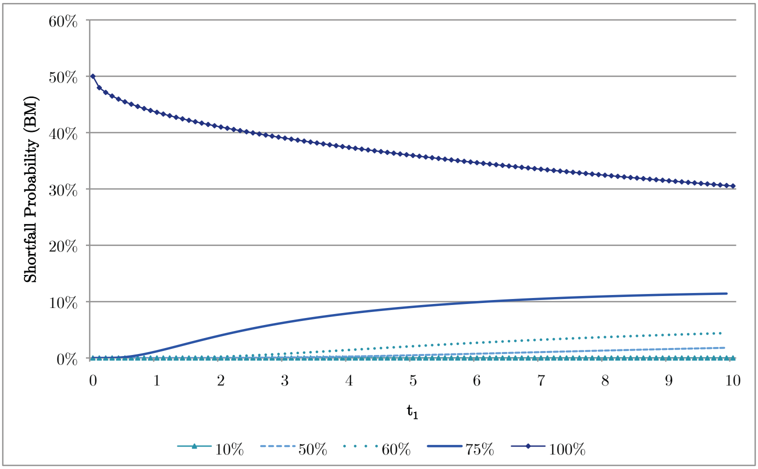


Figure 5.1: Shortfall probability in mixed portfolio (BM)

As expected, the behavior is similar for both definitions of P_{SF} excluding the Private Equity only portfolio and of course again $P_{SF,NBM} < P_{SF,BM}$ holds. It becomes clear from the graphs that the shortfall probability is still very low even for portfolios with a Private Equity share of more than 50% especially when the benchmark is not included. An interesting property is the fact that the shortfall probability is not over the whole interval $t_1 \in [0, T]$ $(T = 10)$ increasing in time (figure 5.1). The explication is similar to the one in section 4.2.2. For portfolios consisting almost completely of Private Equity investments, the benefit from covering the second Private Equity payment $\bar{C}_0$ with the liquid part of the portfolio diminishes and for low values of t_1 (early second payment) drawbacks on the public equity investments have a large weight due to the short recovery period.

Table 5.1 returns the maximal value of the shortfall probability for several levels of Private Equity in the overall portfolio in the border case $t_1 = 10$

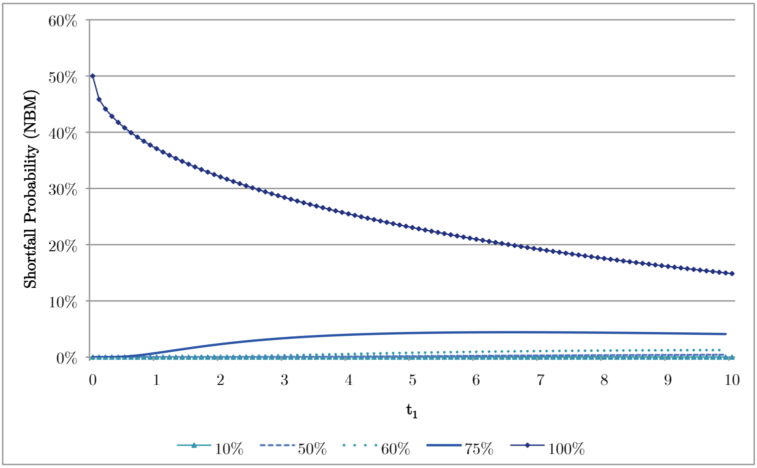


Figure 5.2: Shortfall probability in mixed portfolio (NBM)

and $t_1 = 5$. As the investment phase of a Private Equity fund normally ends after 5-6 years, the probability is very low that after this time a capital call occurs.

	Portfolio Weight (x)	
Shortfall Probability (BM)	$t_1 = 5$	$t_1 = 10$
1%	54%	45%
5%	68%	62%
10%	76%	72%

Table 5.1: Portfolio weight vs. shortfall probability

Table 5.1 shows that a default on commitment becomes only an issue if an investor has the majority of his funds in Private Equity investments. As Private Equity is very illiquid, institutional investors only rarely risk weights larger than 15%-20%. As a result, it can be stated that for most large institutional investors with a weight of Private Equity in the overall portfolio of lower than around 50% the optimal investment of the gap is equal to the analytically derived solution as the shortfall probability approaches zero:

$$\pi^* = \frac{\mu_S - r}{\gamma(1-\delta)\sigma_S^2} - \frac{\delta\sigma_P\rho}{(1-\delta)\sigma_S} \tag{5.20}$$

This finding is shown in chapter 5.4 when the optimal portfolio paths are simulated for different portfolio weights of Private Equity. The reduced shortfall probabilities of a default on commitment when Private Equity is included in a broadly diversified portfolio are a strong rational not to overweight Private Equity. In fact, the vast majority of investors in the *TVE* dataset sticks to weights of Private Equity around 15%-20%. As the opportunity cost depend directly on P_{SF}, opportunity cost are the lower the smaller the weight of Private Equity. For a diversified investor, the analytically derived solution is therefore a general rule to invest the gap. If the risk-free benchmark is not considered, P_{SF} is very low over the whole range during the investment phase and the optimal investment rule is approximately valid even for portfolios consisting entirely of Private Equity.

5.4 Simulation of the Optimal Investment Path

For the derivations in chapter 4 and 5, the crucial assumption was continuous rebalancing meaning that relative portfolio weights do not change over time. Although it was argued before that for large institutional investors it is a realistic assumption, it is often not possible to keep weights exactly constant due to the illiquid nature of Private Equity investments. In order to get an understanding on the goodness of fit of the model derived, the optimal investment path for the gap will be simulated in this section comparing the results to the analytical solutions. Of course, it is clear ex ante that the analytical solutions of the model can only serve as a good proxy for the optimal investment weights when the shortfall probability tends towards zero. In order to execute the simulation, we need to specify the wealth process W_T without the assumption of continuous rebalancing. We assume the bond and equity parts of the overall portfolio to be invested according to the Merton solution. This leads to the following definition of final wealth W_T depending on the time lag of investment t_1

$$
\begin{aligned}
W_T = {} & (1 - P_{SF})[be^{rT} + se^{(\mu_S - \frac{1}{2}\sigma_S^2)T + \sigma_S Z_{S,T}} \\
& + x(1-\delta)e^{(\mu_P - \frac{1}{2}\sigma_P^2)(T-t_1) + \sigma_P(Z_{P,T} - Z_{P,t_1})} + x\delta e^{(\mu_P - \frac{1}{2}\sigma_P^2)T + \sigma_P Z_{P,T}} \\
& + x(1-\delta)(e^{((1-\pi)r + \pi\mu_S - \frac{1}{2}\pi^2\sigma_S^2)t_1 + \pi\sigma_S Z_{S,t_1}} - 1) \\
& e^{((1-\pi_M)r + \pi_M\mu_S - \frac{1}{2}\pi_M^2\sigma_S^2)(T-t_1) + \pi_M\sigma_S(Z_{S,T} - Z_{S,t_1})}] \\
& + P_{SF}[be^{rt_1} + se^{(\mu_S - \frac{1}{2}\sigma_S^2)t_1 + \sigma_S Z_{S,t_1}} + \alpha x\delta e^{(\mu_P - \frac{1}{2}\sigma_P^2)t_1 + \sigma_P Z_{P,t_1}} \\
& + (1-\delta)xe^{((1-\pi)r + \pi\mu_S - \frac{1}{2}\pi^2\sigma_S^2)t_1 + \pi\sigma_S Z_{S,t_1}}] \\
& e^{((1-\pi_M)r + \pi_M\mu_S - \frac{1}{2}\pi_M^2\sigma_S^2)(T-t_1) + \pi_M\sigma_S(Z_{S,T} - Z_{S,t_1})}
\end{aligned}
\tag{5.21}
$$

The corresponding final wealth without time lag of investment and excluding continuous rebalancing is

$$
\bar{W}_T = be^{rT} + se^{(\mu_S - \frac{1}{2}\sigma_S^2)T + \sigma_S Z_{S,T}} + xe^{(\mu_P - p - \frac{1}{2}\sigma_P^2)T + \sigma_P Z_{P,T}} \tag{5.22}
$$

where the return of Private Equity is corrected by opportunity cost p. We can estimate the opportunity cost p using the following expression for the expected utility of final wealth with u(.) being of the type in 3.14 and estimated values marked by a *hat*:

$$
E[u(\hat{\bar{W}}_T)] = E[u(\hat{W}_T)] \tag{5.23}
$$

The optimal investment path π^* is the corresponding value that minimizes p. The following graphs show the simulated values for the optimal investment path depending on the time lag of the second investment for different weights of Private Equity in the portfolio. The values are compared with the Merton solution and the analytical solution derived above.

The parameter values are from table 4.1 besides the correlation ρ between public and private equity which is assumed to be 0.

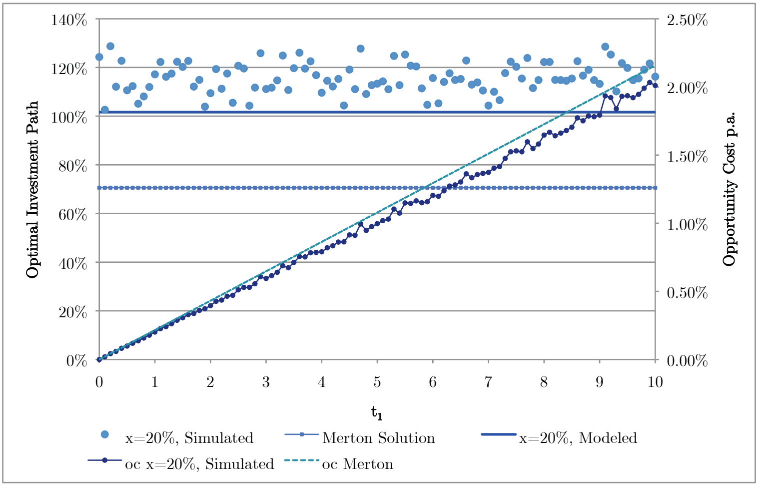


Figure 5.3: Simulation of optimal investment path for $x = 20\%$

Figures 5.3 to 5.6 show the simulated optimal investment path for different weights of Private Equity in a portfolio. The shortfall probability is defined considering the risk-free benchmark. The corresponding simulated opportunity cost are displayed as well, compared to the case where the gap is invested according to the optimal Merton solution from equation 3.12. The graphs show that the model is much closer to the optimal investment weights than the Merton solution and strictly outperforms the opportunity cost for all $t_1 \in [0, T]$. As the model generally overstates opportunity cost, coming from an overstatement of the shortfall probability, it can be expected that the model slightly underweights stocks as P_{SF} is increasing in π. As the overstatement depends on the respective Sharpe ratios of public and private equity, we expect the modeled values to get closer to the simulated ones for less differences in the Sharpe ratio. This flexibility on the characteristics of the Private Equity investment can be seen from figure 5.4. The main advantage of the model compared to the

Merton solution is therefore its sensitivity on SR_P which can be inferred from the graph, as well.

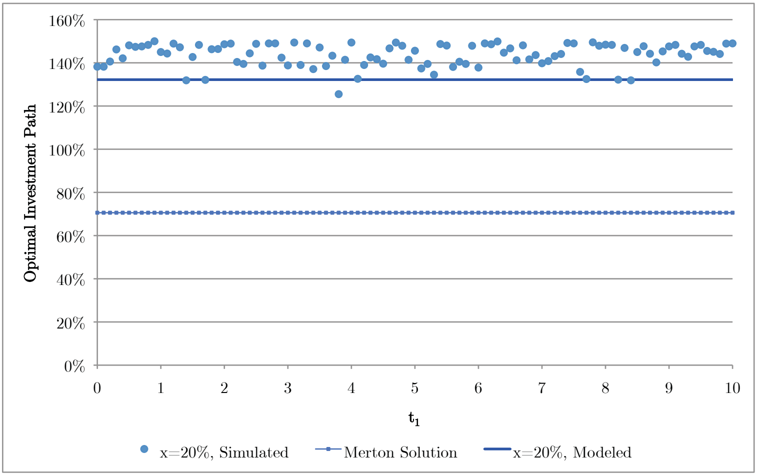


Figure 5.4: Simulation of optimal investment path for $x = 20\%$ and $\mu_P = 16\%$

As figures 5.5 and 5.6 show, the fit of the model to the simulated values is quite good for portfolio weights lower than 50%. The case of a Private Equity-only portfolio is very interesting to look at. The simulated investment path is strictly increasing in t_1 as the return loss is weighted more than the shortfall probability. During the investment phase, the opportunity cost are much less compared to the Merton case. Of course, as the shortfall probability is substantially above zero, the analytically derived values can no more seen as a proxy for optimal portfolio weights in this case.

As a result, it can be concluded that the optimal investment path calculated from the model introduced in the last two chapters strictly outperforms the Merton solution delivering strictly smaller opportunity cost of late investment. Furthermore, it is interesting to see that depending on the parameter values it can be better to invest in a riskier portfolio

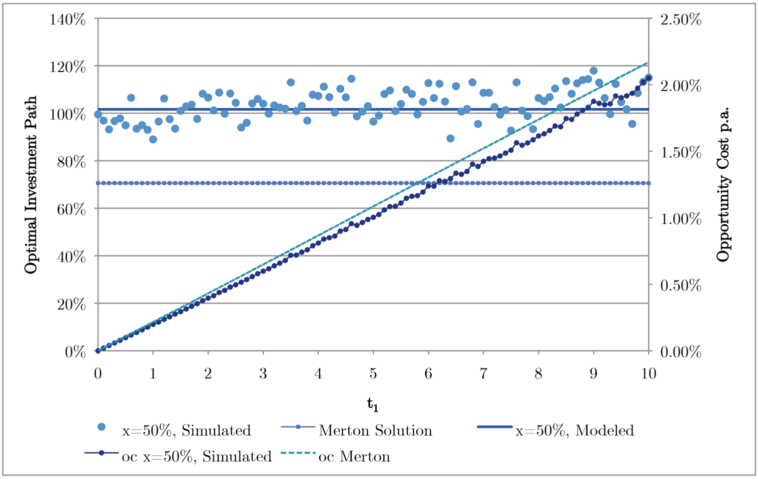


Figure 5.5: Simulation of optimal investment path for $x = 50\%$

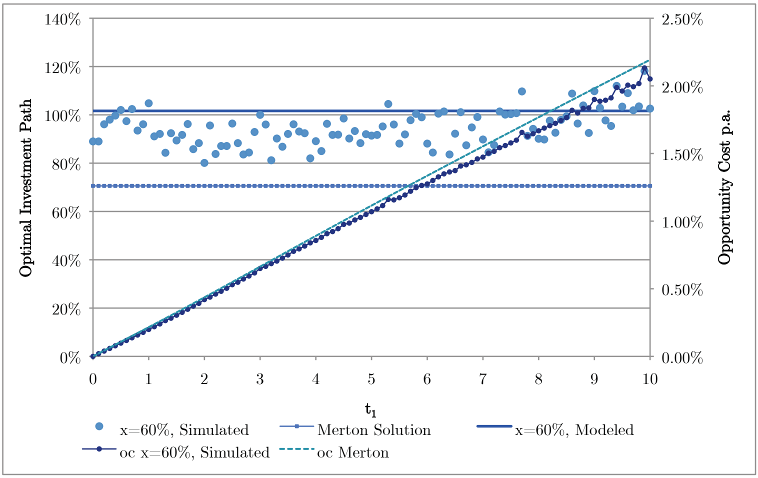


Figure 5.6: Simulation of optimal investment path for $x = 60\%$

than the Merton portfolio. This sensitivity of the model to parameter changes is a major advantage compared to the Merton solution. As the optimal investment path does basically not depend on t_1 for $x < 50\%$, the model even delivers an analytical solution as the shortfall probability can be neglected. Therefore, the model provides the optimal strategy to invest the committed but not yet invested part of the Private Equity portfolio up to portfolio weights of around 50% even for taking the risk-free benchmark into account when P_{SF} is evaluated. As optimal weights of Private Equity are normally less under reasonable parameter assumptions (what will be shown in section 5.6), the model can serve as a good rule to invest.

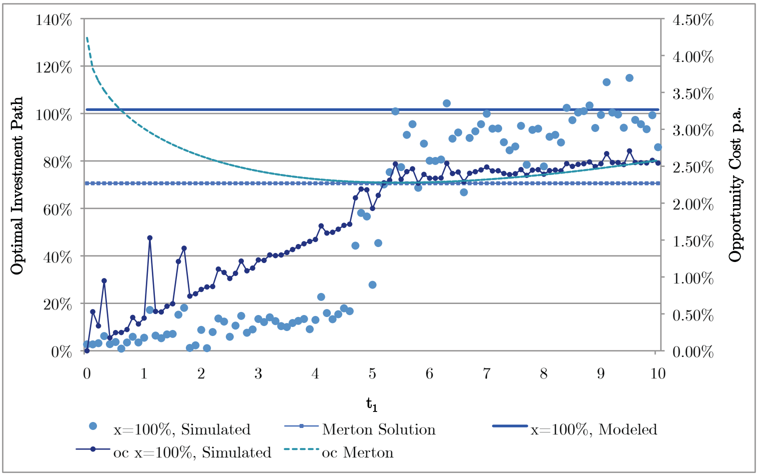


Figure 5.7: Simulation of optimal investment path for $x = 100\%$

5.5 Liquidity Constraints

5.5.1 Funds with Fixed Payout Rates

It is obvious that constraints on the liquid part of a portfolio have an influence on the funds available to prevent a potential shortfall in t_1 and therefore on P_{SF} itself. The first liquidity constraint of an investor's portfolio we want to look at are fixed payout rates. An institutional asset manager with a large number of individual investors, like pension or endowment funds etc., is normally faced with certain annual payouts. In this section it is assumed that a fixed fraction c of initial wealth W_0 has to be paid out annually. In order to calculate the shortfall probability, we assume a virtual account in which the time value of all payouts over the investment horizon $[0,T]$ is collected. This account cannot be considered to prevent a default on commitment. As the investor has to pay out a fix fraction each period, he risks bankruptcy if he does not keep those payouts in a separate account. As a result, the investor is faced with a virtual reduction of initial wealth. Considering the payout rates over the whole investment horizon $t \in [0,T]$, we get a total amount of

$$\int_0^T cW_0 e^{-rt} dt = W_0 \frac{c}{r}(1 - e^{-rT}) \tag{5.24}$$

by which initial wealth is reduced. Again, the Private Equity investment is split in two parts and δC_0 is invested at $t_0 = 0$. But as the annual payout has to be financed from the liquid part of the portfolio, remaining liquid wealth now reduces to

$$\begin{aligned} \tilde{W_0} &= \bar{W_0} - \underbrace{W_0 \frac{c}{r}(1 - e^{-rT})}_{k} = (1-\delta)C_0 + S_0 + B_0 - (C_0 + S_0 + B_0)k \\ &= (1-\delta-k)C_0 + (1-k)S_0 + (1-k)B_0 \end{aligned} \tag{5.25}$$

This changes the respective portfolio weights for the liquid portfolio part to

$$\tilde{x} = \frac{(1-\delta-k)C_0}{(1-\delta-k)C_0 + (1-k)S_0 + (1-k)B_0} = \frac{\tilde{C_0}}{\tilde{C_0} + \tilde{S_0} + \tilde{B_0}} \tag{5.26}$$

$$\tilde{s} = \frac{(1-k)S_0}{(1-\delta-k)C_0 + (1-k)S_0 + (1-k)B_0} = \frac{\tilde{S_0}}{\tilde{C_0} + \tilde{S_0} + \tilde{B_0}} \tag{5.27}$$

$$\tilde{b} = \frac{(1-k)B_0}{(1-\delta-k)C_0 + (1-k)S_0 + (1-k)B_0} = \frac{\tilde{B_0}}{\tilde{C_0} + \tilde{S_0} + \tilde{B_0}} \tag{5.28}$$

summing up to one. The respective wealth processes for the three asset classes remain the same as before. The derivation of the shortfall probability is now completely analogous to section 5.2. Again, a shortfall occurs if liquid assets do not cover the second investment into the Private Equity fund, $\bar{C_0} = (1-\delta)C_0$:

$$\tilde{C_{t_1}} + \tilde{S_{t_1}} + \tilde{B_{t_1}} < \bar{C_0}e^{rt_1} \tag{5.29}$$

The left hand side corresponds to the reduced liquid wealth given by the weights above. Dividing again by liquid wealth, $\tilde{W_0}$, and taking logs leads to

$$\begin{aligned} \frac{\tilde{C_{t_1}} + \tilde{S_{t_1}} + \tilde{B_{t_1}}}{\tilde{W_0}} &< \frac{\bar{C_0}}{\tilde{W_0}}e^{rt_1} = ye^{rt_1} \\ ln(\frac{\tilde{C_{t_1}} + \tilde{S_{t_1}} + \tilde{B_{t_1}}}{\tilde{W_0}}) &= ln(\tilde{W_{t_1}}) < lny + rt_1 \end{aligned} \tag{5.30}$$

The dynamics are calculated in the same way than in section 5.2 using Itô's Lemma leading to the same wealth process $dln(W_t)$ but with the weights from equations 5.26 to 5.28.

Recall from equation 5.18

$$\frac{Z_{S,t_1}}{\sqrt{t_1}} < \frac{ln\bar{x}}{(\pi\bar{x}+\bar{s})\sigma_S\sqrt{t_1}} - \frac{[\mu_S - r - \frac{1}{2}\sigma_S^2(\pi\bar{x}+\bar{s})]t_1}{\sigma_S\sqrt{t_1}} = z \tag{5.31}$$

Changing the portfolio weights and the expression on the right hand side of 5.30 leads to

$$\frac{Z_{S,t_1}}{\sqrt{t_1}} < \frac{lny}{(\pi\tilde{x}+\tilde{s})\sigma_S\sqrt{t_1}} - \frac{[\mu_S - r - \frac{1}{2}\sigma_S^2(\pi\tilde{x}+\tilde{s})]t_1}{\sigma_S\sqrt{t_1}} = z \tag{5.32}$$

and

$$P_{SF,BM}(\bar{C_{t_1}} + S_{t_1} + B_{t_1} < \bar{C_0}e^{rt_1}) = \Phi(z) \tag{5.33}$$

The following figures show the shortfall probability for $c = 1\%$, 3%, 5% and 10%.

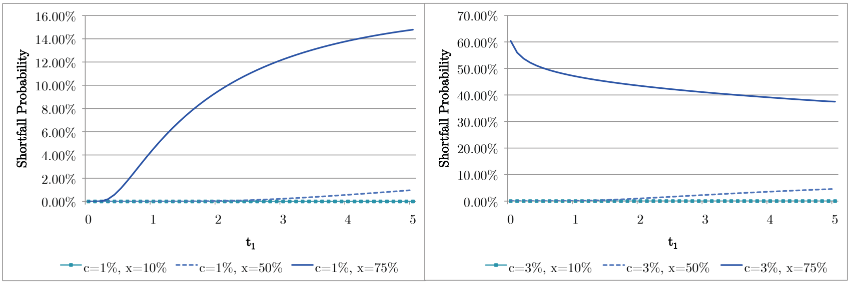


Figure 5.8: Shortfall probability with fixed annual payment (1)

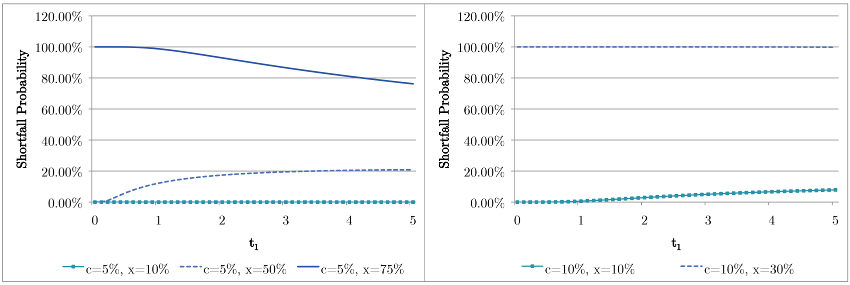


Figure 5.9: Shortfall probability with fixed annual payment (2)

The picture is quite different from what we had in section 5.2. For $c = 1\% - 5\%$ the shortfall probability is still negligible for a weight of Private Equity around 10%. But for larger values, the shortfall probability starts to increase substantially. The shortfall probability is already very large for $c = 5\%$ and $x = 50\%$. The illiquidity of Private Equity investments can be seen clearly from the graphs, as the shortfall probabilities rise sharply with growing portfolio weights of Private Equity. For $x = 75\%$, it can be seen that the shortfall probability is not always increasing in t_1. This makes intuitively sense, as for large weights of Private Equity the shortfall probability decreases if the second investment into the fund comes later and more liquid funds are available to cover a potential default on commitment. This evaluation shows clearly that for investors facing fixed annual payout rates, the share of Private Equity should not exceed $x = 10\% - 15\%$ for payout rates as large as 5% if the shortfall probability wants to be kept below 1%. The vast majority of the diversified portfolios in the *TVE* database sticks to these bounds.

5.5.2 Funds with Illiquid Portfolio Parts

For several reasons, it might be possible that an investor is restricted by some liquidity requirement. Let us assume that an investor needs a fraction a of total wealth to be held in liquid assets (cash equivalents or very liquid shares). If we express this need as a fraction h of shares, the available funds to cover a second call on commitment are reduced by this part. Using $\tilde{W_0} = \bar{C_0} + (1-h)S_0 + B_0$, obviously the shortfall probability in 5.19 changes to[44]

$$\begin{aligned}
&P_{SF}(\bar{C_{t_1}} + (1-h)S_{t_1} + B_{t_1} < \bar{C_0}e^{rt_1}) \\
&= P_{SF}(\frac{\bar{C_{t_1}} + (1-h)S_{t_1} + B_{t_1}}{\tilde{W_0}} < \tilde{x}e^{rt_1}) \\
&= P_{SF}(ln(\frac{\bar{C_{t_1}} + (1-h)S_{t_1} + B_{t_1}}{\tilde{W_0}}) < ln\tilde{x} + rt_1) \\
&= P_{SF}(\frac{Z_{S,t_1}}{\sqrt{t_1}} < \frac{ln\tilde{x}}{(\pi\tilde{x} + \tilde{s})\sigma_S\sqrt{t_1}} - \frac{[\mu_S - r - \frac{1}{2}\sigma_S^2(\pi\tilde{x} + \tilde{s})]t_1}{\sigma_S\sqrt{t_1}})
\end{aligned} \quad (5.34)$$

with

$$h = \frac{a}{S_{t_1}}(C_{t_1} + S_{t_1} + B_{t_1}) = \frac{a}{s} \quad (5.35)$$

and

$$\tilde{x} = \frac{\bar{C_t}}{\bar{C_t} + (1-h)S_t + B_{t_1}} = \frac{\bar{x}}{\bar{x} + (1-h)\bar{s} + \bar{b}}$$

[44]The calculations are similar to the ones in section 5.2 and Appendix C.5.

$$\tilde{s} = \frac{(1-h)S_t}{\bar{C}_t + (1-h)S_t + B_{t_1}} = \frac{(1-h)\bar{s}}{\bar{x} + (1-h)\bar{s} + \bar{b}} \tag{5.36}$$

$$\tilde{b} = \frac{B_t}{\bar{C}_t + (1-h)S_t + B_{t_1}} = \frac{\bar{b}}{\bar{x} + (1-h)\bar{s} + \bar{b}}$$

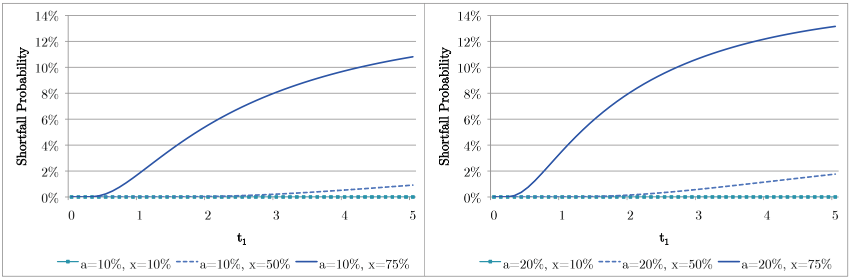


Figure 5.10: Shortfall probability with constant liquidity need

Figure 5.10 shows the shortfall probability for constant levels of liquidity. As one would expect, P_{SF} is increasing in t_1 and is larger for higher levels of Private Equity x when a smaller liquid portfolio part must finance a constant level of liquidity. On the other hand, it is very surprising that the magnitude of the effect which has a constant liquidity requirement on the shortfall probability is only minor. On the left hand side of figure 5.10 the required level of liquidity is $a = 10\%$ of overall wealth and on the right hand side $a = 20\%$. The effects on the shortfall probability are modest. In the base case without liquidity need (see chapter 5.2) the shortfall probability for $t_1 = 5$ and $x = 75\%$ is 9.19%, only about 1.5% lower than with a 10% liquidity requirement. For small weights of Private Equity in the portfolio (i.e. $x = 10\% - 20\%$) the relative increases in P_{SF} are substantial when the liquidity need is increased but the absolute influence is still negligible and P_{SF} close to zero. As a result, the optimal solution derived in section 5.3 remains valid. The higher the level of Private Equity the lower is the sensitivity of P_{SF} towards changes in the liquidity need. For π the optimal amount derived earlier is chosen. It is easy to see from equation 5.34 that P_{SF} is very sensitive to changes in π. Obviously, if P_{SF} is larger than zero, the optimal level of π has to be evaluated using simulation techniques analogously to chapter 5.4 in order to evaluate the optimal value for π. If π is chosen close to zero, the shortfall probability tends towards zero even for a level of liquidity of $a = 20\%$ and $x = 75\%$. As a result, the opportunity cost strongly depend on the level of required liquidity and are very sensitive to changes of it.

5.6 Portfolio Optimization Adapted to Commitment Calls

5.6.1 Derivation and Interpretation of the Optimal Portfolio Weights

In this section a portfolio optimization model is derived taking the timing of the investment flow into a Private Equity investment (and therefore the related opportunity cost) into account. As was shown in chapter 3.3.3, the optimal portfolio weights are constant in time. As the goal is now to optimize the portfolio consisting of Private Equity, risky assets and bonds taking the delayed investment path into the Private Equity fund into consideration, the portfolio has to be optimized over both time periods $\tau_1 = [0, t_1]$ and $\tau_2 = [t_1, T]$. We fix the investor's investment horizon to T, the lifetime of the Private Equity fund. It is obvious that the portfolio weights will change over the two time periods. Within the two time periods the portfolio weights are again constant following the same rationale as in chapter 3.3.3. As Private Equity investments are (almost) illiquid, their weights have to be fixed over both periods. Therefore, we have to optimize the weight of the Private Equity investment (x), the weights of the risky asset in τ_1 (s_1) and τ_2 (s_2) and the weights of the risk-free investments in both periods (b_1, b_2) in order to maximize an investor's expected utility of final wealth. It is still assumed that the investment into the Private Equity fund is split in two parts: an initial payment δC_0 at time $t_0 = 0$ and the remainder at t_1. The respective weights of Private Equity are therefore δx and x. The commitment C_0 is again normalized to 1 and the portfolio is rebalanced continuously. The assets' dynamics are equivalent to the expressions in 3.21 to 3.23. The respective weights are given by

$$
\begin{aligned}
t \in \tau_1 : \delta x &= \frac{C_t}{W_t} = \frac{C_t}{C_t + S_t + B_t} \\
s_1 &= \frac{S_t}{W_t} \\
b_1 &= \frac{B_t}{W_t} = 1 - \delta x - s_1
\end{aligned} \tag{5.37}
$$

$$
\begin{aligned}
t \in \tau_2 : \ x &= \frac{C_t}{W_t} = \frac{C_t}{C_t + S_t + B_t} \\
s_2 &= \frac{S_t}{W_t} \\
b_2 &= \frac{B_t}{W_t} = 1 - x - s_2
\end{aligned} \tag{5.38}
$$

The log-wealth at the end of both time periods therefore is

$$t_1 : \qquad ln(W_{t_1}) = ln(C_{t_1} + S_{t_1} + B_{t_1}) \tag{5.39}$$

$$T : \qquad ln(W_T) = ln(C_T + S_T + B_T) \tag{5.40}$$

As a result, the dynamics of $\mathrm{d}ln(W_{t_1})$ and $\mathrm{d}ln(W_T)$ can again be derived using Itô's Lemma analogously to equations 3.27 and 5.13 and the resulting process is

$$
\begin{aligned}
t \in \tau_1 : \mathrm{d}ln(W_t) = \delta x \frac{dC_t}{C_t} + s_1 \frac{dS_t}{S_t} + (1 - s_1 - \delta x)\frac{dB_t}{B_t} - \frac{1}{2}\delta^2 x^2 (\frac{dC_t}{C_t})^2 \\
- \frac{1}{2} s_1^2 (\frac{dS_t}{S_t})^2 - \delta x s_1 \frac{dC_t}{C_t}\frac{dS_t}{S_t}
\end{aligned} \tag{5.41}
$$

Using Itô calculus and substituting the respective differential equations yields the wealth process in $t \in \tau_1$

$$t \in \tau_1 : \mathrm{d}ln(W_t) = (\delta x \mu_P + s_1 \mu_S + (1 - s_1 - \delta x)r - \frac{1}{2}\delta^2 x^2 \sigma_P^2 - \frac{1}{2} s_1^2 \sigma_S^2 - \delta x s_1 \sigma_P \sigma_S \rho)dt + \delta x \sigma_P dZ_{P,t} + s_1 \sigma_S dZ_{S,t} \tag{5.42}$$

and analogously in $t \in \tau_2$ for $\mathrm{d}ln(W_t)$ only replacing the respective portfolio weights ($\delta x \rightarrow x$, $s_1 \rightarrow s_2$ and $(1 - \delta x - s_1) \rightarrow (1 - x - s_2)$). From $ln(W_{t_1}) - ln(W_0)$ and $ln(W_T) - ln(W_{t_1})$ final log-wealth $ln(W_T)$ can be calculated in a straightforward way assuming again that $W_0 = 1$:

$$\begin{aligned} lnW_T = \\ &(x\mu_P + s_2\mu_S + (1 - s_2 - x)r - \frac{1}{2}x^2\sigma_P^2 - \frac{1}{2}s_2^2\sigma_S^2 - xs_2\sigma_P\sigma_S\rho)(T - t_1) \\ &+ x\sigma_P(Z_{P,T} - Z_{P,t_1}) + s_2\sigma_S(Z_{S,T} - Z_{S,t_1}) \\ &+ (\delta x\mu_P + s_1\mu_S + (1 - s_1 - \delta x)r - \frac{1}{2}\delta^2x^2\sigma_P^2 - \frac{1}{2}s_1^2\sigma_S^2 - \delta xs_1\sigma_P\sigma_S\rho)t_1 \\ &+ \delta x\sigma_P Z_{P,t_1} + s_1\sigma_S Z_{S,t_1} \end{aligned} \tag{5.43}$$

Rearranging and exponentiating leads to final wealth:

$$\begin{aligned} W_T = \\ &\exp[(x\mu_P + s_2\mu_S + (1 - s_2 - x)r - \frac{1}{2}x^2\sigma_P^2 - \frac{1}{2}s_2^2\sigma_S^2 - xs_2\sigma_P\sigma_S\rho)T \\ &- ((1 - \delta)x\mu_P + (s_2 - s_1)\mu_S + (s_1 - s_2 - (1 - \delta)x)r - \frac{1}{2}(1 - \delta^2)x^2\sigma_P^2 \\ &- \frac{1}{2}(s_2^2 - s_1^2)\sigma_S^2 - (s_2 - \delta s_1)x\sigma_P\sigma_S\rho)t_1 + x\sigma_P(Z_{P,T} - Z_{P,t_1}) \\ &+ s_2\sigma_S(Z_{S,T} - Z_{S,t_1}) + \delta x\sigma_P Z_{P,t_1} + s_1\sigma_S Z_{S,t_1}] \end{aligned} \tag{5.44}$$

Taking expectations of the utility[45] of final wealth leads to[46]

[45] The utility function is again of CRRA-type as in equation 3.14.
[46] For detailed derivations the reader is referred to Appendix C.6.

$$
\begin{aligned}
E[u(W_T)] = &\frac{1}{1-\gamma}\exp[(1-\gamma)(x\mu_P + s_2\mu_S + (1-s_2-x)r - \frac{1}{2}x^2\sigma_P^2 \\
&- \frac{1}{2}s_2^2\sigma_S^2 - xs_2\sigma_P\sigma_S\rho)T - (1-\gamma)((1-\delta)x\mu_P + (s_2-s_1)\mu_S \\
&+ (s_1 - s_2 - (1-\delta)x)r - \frac{1}{2}(1-\delta^2)x^2\sigma_P^2 - \frac{1}{2}(s_2^2 - s_1^2)\sigma_S^2 \\
&- (s_2 - \delta s_1)x\sigma_P\sigma_S\rho)t_1 + \frac{1}{2}(1-\gamma)^2((x^2\sigma_P^2 + s_2^2\sigma_S^2 + 2xs_2\sigma_P\sigma_S\rho)T \\
&- (1-\delta^2)x^2\sigma_P^2 t_1 - (s_2^2 - s_1^2)\sigma_S^2 t_1 - 2x(s_2 - \delta s_1)\sigma_P\sigma_S\rho t_1)]
\end{aligned}
\tag{5.45}
$$

In order to derive the optimal portfolio weights over the two time periods, the expression in 5.45 has to be maximized. Again, the problem can be reduced to maximizing the exponent divided by $1-\gamma$. If we introduce $F(x, s_1, s_2)$ as the function to be optimized the problem reads

$$
\max_{x,s_1,s_2} E[u(W_T)] \hat{=} \max_{x,s_1,s_2} F(x, s_1, s_2) \tag{5.46}
$$

with

$$
\begin{aligned}
&F(x, s_1, s_2) = \\
&[(x\mu_P + s_2\mu_S + (1-s_2-x)r - \frac{1}{2}\gamma x^2\sigma_P^2 - \frac{1}{2}\gamma s_2^2\sigma_S^2 - \gamma xs_2\sigma_P\sigma_S\rho)T \\
&- ((1-\delta)x\mu_P + (s_2-s_1)\mu_S + (s_1 - s_2 - (1-\delta)x)r - \frac{1}{2}\gamma(1-\delta^2)x^2\sigma_P^2 \\
&- \frac{1}{2}\gamma(s_2^2 - s_1^2)\sigma_S^2 - \gamma(s_2 - \delta s_1)x\sigma_P\sigma_S\rho)t_1]
\end{aligned}
$$

Using matrix notation, the problem can be simplified to

$$
\max_{\boldsymbol{w}} E[u(W_T)] \hat{=} \max_{\boldsymbol{w}} F(\boldsymbol{w}) \tag{5.47}
$$

with

$$F(\boldsymbol{w}) = \boldsymbol{w\mu} - \frac{1}{2}\gamma \boldsymbol{w^T} V \boldsymbol{w}$$

and $\boldsymbol{w}$ being the optimal optimal portfolio weights, $\boldsymbol{\mu}$ the respective time-weighted risk premia and V the covariance matrix:

$$V = \begin{pmatrix} \sigma_P^2(T-(1-\delta^2)t_1) & \sigma_P\sigma_S\rho(T-t_1) & \delta\sigma_P\sigma_S\rho t_1 \\ \sigma_P\sigma_S\rho(T-t_1) & \sigma_S^2(T-t_1) & 0 \\ \delta\sigma_P\sigma_S\rho t_1 & 0 & \sigma_S^2 t_1 \end{pmatrix};$$
$$\boldsymbol{w} = \begin{pmatrix} x \\ s_2 \\ s_1 \end{pmatrix}; \boldsymbol{\mu} = \begin{pmatrix} (\mu_P - r)(T-(1-\delta)t_1) \\ (\mu_S - r)(T-t_1) \\ (\mu_S - r)t_1 \end{pmatrix} \tag{5.48}$$

Deriving the first order conditions (FOC) from 5.47 is straightforward

$$\gamma V \boldsymbol{w} = \boldsymbol{\mu} \tag{5.49}$$

and the optimal weights therefore are

$$\boldsymbol{w}^* = \frac{1}{\gamma} V^{-1} \boldsymbol{\mu} \tag{5.50}$$

Again, a solution only exists if the determinant of the covariance matrix is not equal to zero. This leads to the condition $|\rho| \neq 1$. Since V is a regular covariance matrix, it is positive definite. As a result, the optimal solutions derived above always lead to a maximum. Using Cramer's rule leads to the optimal weights[47]

$$x^* = \frac{(\mu_P - r - \frac{\mu_S - r}{\sigma_S}\sigma_P\rho)(T-(1-\delta)t_1)}{\gamma\sigma_P^2(1-\rho^2)(T-(1-\delta^2)t_1)} \tag{5.51}$$

[47] For detailed derivations the reader is referred to Appendix C.7.

$$s_1^* = \frac{(\mu_S - r)\frac{\sigma_P^2}{\sigma_S^2}(T - (1-\delta^2)t_1 - \rho^2(1-\delta)(T-t_1))}{\gamma\sigma_P^2(1-\rho^2)(T-(1-\delta^2)t_1)} - \frac{(\mu_P - r)\frac{\sigma_P}{\sigma_S}\delta\rho(T-(1-\delta)t_1)}{\gamma\sigma_P^2(1-\rho^2)(T-(1-\delta^2)t_1)} \quad (5.52)$$

$$s_2^* = \frac{(\mu_S - r)\frac{\sigma_P^2}{\sigma_S^2}(T-(1-\delta^2)t_1 + \delta(1-\delta)\rho^2 t_1) - (\mu_P - r)\frac{\sigma_P}{\sigma_S}\rho(T-(1-\delta)t_1)}{\gamma\sigma_P^2(1-\rho^2)(T-(1-\delta^2)t_1)} \quad (5.53)$$

Expressing s_1^* and s_2^* in terms of the optimal weight of Private Equity, x^*, and deriving the optimal weights of the risk-free asset leads to

$$x^* = \frac{(SR_P - SR_S\rho)(T-(1-\delta)t_1)}{\gamma\sigma_P(1-\rho^2)(T-(1-\delta^2)t_1)} \quad (5.54)$$

$$s_1^* = \frac{\mu_S - r}{\gamma\sigma_S^2} - \frac{\sigma_P}{\sigma_S}\delta\rho x^* = \pi_M - \frac{\sigma_P}{\sigma_S}\delta\rho x^* \quad (5.55)$$

$$s_2^* = \frac{\mu_S - r}{\gamma\sigma_S^2} - \frac{\sigma_P}{\sigma_S}\rho x^* = \pi_M - \frac{\sigma_P}{\sigma_S}\rho x^* \quad (5.56)$$

$$b_1^* = 1 - s_1^* - \delta x^* = 1 - \pi_M - (1 - \frac{\sigma_P}{\sigma_S}\rho)\delta x^* \quad (5.57)$$

$$b_2^* = 1 - s_2^* - x^* = 1 - \pi_M - (1 - \frac{\sigma_P}{\sigma_S}\rho)x^* \quad (5.58)$$

The optimal weight of the Private Equity investment depends on the risk adjusted premia (Sharpe ratio) of private and public equity, their correlation and the investor's coefficient of risk aversion. Additionally, in contrast to the results in chapter 3.3.3 where the delayed investment flow is not taken into account, it also depends on the investor's investment horizon T, the timing of the second investment round of the Private Equity fund (t_1) and the respective amount of funds invested (δ). This has interesting implications for the selection of Private Equity funds: as the optimal investment weights have to be rebalanced over the two time periods and depend on the investment path, it is important to invest in funds with a reliable general management that can stick to a predefined investment path. Therefore, following the optimal investment path cannot be controlled completely by the investor but depends also on the fund management. The condition which must hold in order to provide an incentive for an investor to invest in Private Equity at all is

$$x^* \geq 0 \Longrightarrow SR_P \geq SR_S\rho \tag{5.59}$$

stating that the Sharpe ratio of Private Equity has to be larger than the correlation weighted one of public equity.

It can be seen easily from equations 5.55 and 5.56 that $s_1 > s_2$ holds if $\rho > 0$ (and $x^* > 0$) and is never equal over both time periods if there is an investment gap for $\rho \neq 0$. This is somewhat surprising as intuitively one could expect that for low levels of positive or negative correlation parts of the gap are invested in the risky asset in τ_1 and redirected into the Private Equity fund in τ_2 in order to offset parts of the foregone profit due to the delayed investment. The explanation are again diversification considerations. When the correlation is negative, the public asset provides a good hedge against downturns in the Private Equity investment and the share of Private Equity of the overall portfolio becomes quite large (as will be shown later) increasing the risks considerably especially in τ_2 when the investor's commitment is fully invested which can be diversified by the risky asset. If the total investment in both risky asset classes is compared over the two time periods, the result depends again on the correlation structure and the following condition can be derived from equations 5.55 to 5.58 if the risky assets are assumed to be larger in τ_1 than in τ_2:

$$s_1 + \delta x^* > s_2 + x^* \Longrightarrow \rho > \frac{\sigma_S}{\sigma_P} \tag{5.60}$$

Therefore, only for rather high levels of correlation total risky assets are larger in τ_1 than in τ_2. For $\rho \leq 0$ total risky assets are always larger in the second time period. For $\rho = \frac{\sigma_S}{\sigma_P}$ Private Equity is a perfect substitute for public equity and in case of $\rho = 0$ for the risk-free investment (liquidity). This finding is equivalent to the findings in chapter 3.3.3 without taking the delayed investment path into consideration.

The investment weights of the public equity over the two time periods differ only in the weights of Private Equity that are deducted. It can be seen directly that the weights in both cases are only larger than the Merton solution in case of negative correlation which is equal to the results without investment gap. For $\rho = 0$ it further holds that the weight of public equity is equal in both time periods and furthermore equal to the weight of the risky asset in case of no investment gap ($s_1^* = s_2^* = s_{NG}^*$). Therefore, in case of negative correlation the relation is $s_1^* < s_{NG}^* < s_2^*$ and for $\rho > 0$ we have $s_1^* > s_{NG}^* > s_2^*$.

Comparing the optimal weights of Private Equity for the two models with and without investment gap, the following relation is observed:

$$x^* = \frac{T - (1-\delta)t_1}{T - (1-\delta^2)t_1} x_{NG}^* \tag{5.61}$$

The fraction on the right-hand side is always larger than one if $\delta > 0$ holds. As there would be no investment delay for $\delta = 0$, $x^* > x_{NG}^*$ always holds and the optimal weight of Private Equity in the portfolio is always strictly larger when taking the delayed investment path into account. The goal is to offset parts of the forgone returns due to the time-lag of investment into Private Equity funds.

The dynamics of the optimal weight of Private Equity depending on the correlation structure ($\frac{dx^*}{d\rho}$) remain the same as in the case without investment delay and depends on the respective Sharpe ratios of private and public equity. Overall, the optimal weights are very sensitive on the correlation ρ. As a result, it is very important to estimate the correlation structure carefully in order to derive the optimal investment path. The dynamics depending on the fraction of commitment invested at the inception of the fund, δ, depend on the relative timing of the second payment t_1 compared to the overall investment horizon T.

The relation of the optimal investment weights and the optimal solution derived in chapter 4, equation 4.40, can also be shown. The weight of public equity in the first time period, τ_1 must be equal to the optimal weight of public shares on the portfolio part excluding Private Equity plus the amount of shares from committed but not yet invested capital. As the weighting between stocks and bonds follows the Merton portfolio rule, π_M, and the optimal investment of the gap follows the optimal rule derived in chapter 4, the following condition must hold:

$$s_1 \stackrel{!}{=} (1 - x^*)\pi_M + (1 - \delta)x^*\pi^* \tag{5.62}$$

Proof:

Recall that, according to equation 4.40, π^* follows

$$\pi^*_{SF} = \frac{\mu_S - r - \gamma\delta\sigma_S\sigma_P\rho}{\gamma(1-\delta)\sigma_S^2} = \frac{1}{1-\delta}(\pi_M - \frac{\sigma_P}{\sigma_S}\rho\delta) = \pi^*_{NSF}$$

This leads to

$$\begin{aligned}(1 - x^*)\pi_M + (1 - \delta)x^*\pi^* &= (1 - x^*)\pi_M + (1 - \delta)x^*\frac{1}{1-\delta}(\pi_M - \frac{\sigma_P}{\sigma_S}\rho\delta) \\ &= \pi_M - \frac{\sigma_P}{\sigma_S}\rho\delta x^* = s_1\end{aligned}$$

which is equal to condition 5.62 and concludes the proof and shows that in case the investor does not include Private Equity to the portfolio the Merton optimization rule optimizes his expected utility of wealth.

5.6.2 Illustration and Sensitivity of the Optimal Portfolio Weights

This section shows the dynamics of the optimal portfolio weights. All figures assume the common parameter values if not stated else and a coefficient of risk aversion of $\gamma = 5.0$. Most studies find a coefficient of risk aversion between 5 and 10 reasonable for institutional investors[48]. Figure 5.11 shows the optimal weights of all asset classes for both time

[48] The absolute value of risk aversion for different types of investors is very controversially discussed in the literature. A good overview on the literature is given in Janecek (2004).

periods and $\rho = 0.5$. As the correlation is close to $\frac{\sigma_S}{\sigma_P} = 0.61$, Private Equity is almost a perfect substitute for public equity. This can be seen graphically as the risk-free weight does basically not change over both time periods. If the correlation is changed to $\rho = 0$ (Figure 5.12), the optimal amount of Private Equity does basically not change for different t_1. But now, Private Equity is a perfect substitute for liquidity and the weight of public assets is constant over time. For negative correlation (Figure 5.13) the total of risky asset in τ_2 is considerably larger than in τ_1. The risk-free asset is even shorted in the second time period and not only Private Equity but also public equity is almost a perfect substitute for liquidity as the increase in both risky assets in τ_2 comes almost completely at the expense of the risk-free asset.

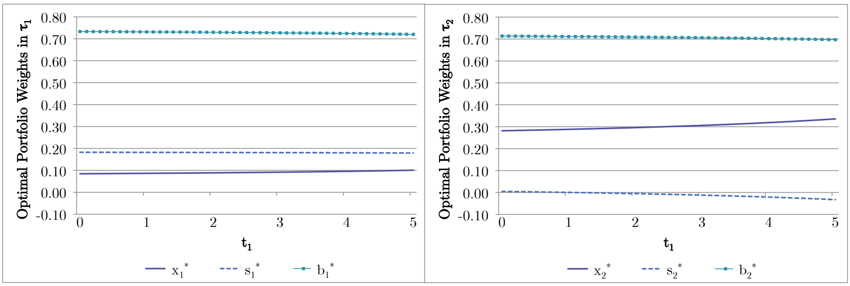


Figure 5.11: Optimal portfolio weights in τ_1 and τ_2, $\rho = 0.5$

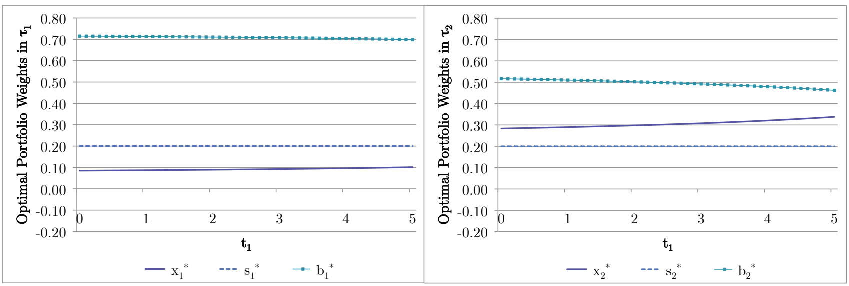


Figure 5.12: Optimal portfolio weights in τ_1 and τ_2, $\rho = 0$

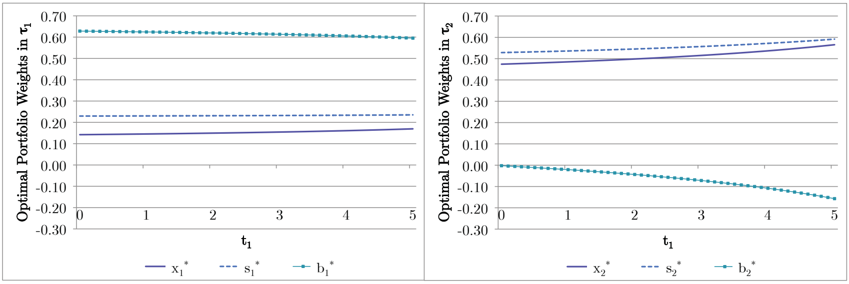


Figure 5.13: Optimal portfolio weights in τ_1 and τ_2, $\rho = -0.5$

Figure 5.14 shows the sensitivity of the optimal weights on changes in the correlation structure. The u-shaped form of x^* found in chapter 5.6.1 can be seen clearly. The sensitivity around 0 is very low. Intuitively it is somewhat puzzling that for low correlation the optimal weights of Private Equity are lower than for higher positive or negative correlation attaining its minimum for $\rho = 0.27$ in both time periods. The explanation is that the diversification potential of public equity with respect to Private Equity outweighs diversification benefits from lower correlation which is an interesting finding.

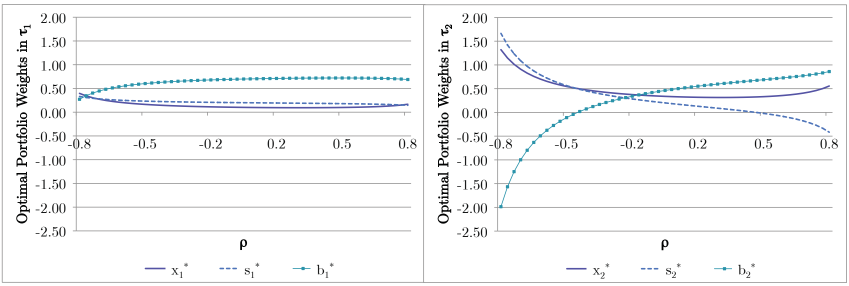


Figure 5.14: Optimal portfolio weights in τ_1 and τ_2, $t_1 = 5$

When the proportion of commitment paid in at the beginning of the fund increases, the weight of Private Equity increases as well in τ_1 (Figure 5.15). This is intuitively clear as the desired fraction of Private Equity in the overall portfolio can be attained earlier. In τ_2, the fraction of Private Equity reflects the weights an investor has initially dedicated to Private Equity as this weight cannot be changed over time and is only different from the one in τ_1 due to the delayed investment path. The sensitivities in the second time period when the investor is fully invested in Private Equity towards δ are rather low showing an inversely u-shaped pattern. Intuitively, one would expect a decreasing pattern as the more can be invested earlier in the Private Equity fund the lower has the amount of "overcommitment" to be compared to the weights without a delayed investment path. The reason are diversification benefits especially in the first period.

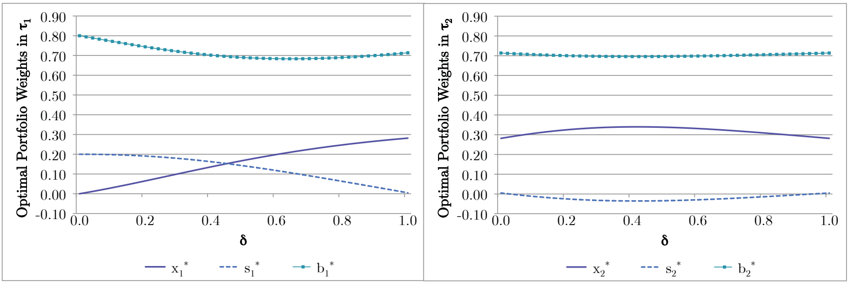


Figure 5.15: Optimal portfolio weights in τ_1 and τ_2, $t_1 = 5$, $\rho = 0.5$

Table 5.2 shows the optimal portfolio weights for Private Equity, stocks and bonds according to the two models with and without considering late investment for different levels of risk aversion and two different points in time when the second investment into Private Equity occurs. The differences between the two models are substantial and especially during the first time period, τ_1, the assumption that committed capital is equal to invested capital leads to an investment path that is far from being optimal. If the second call comes later, the differences become even larger for lower levels of γ. In general, the higher the level of risk aversion the lower the differences between the models. An interesting finding is the fact that the share of Private Equity in total risky assets remains constant for both models and over both time periods for different levels of risk aversion although the risk profiles differ substantially. In τ_2, when the investor is fully invested, x_{NG} is generally too low with a constant proportion over different γ.

It is also interesting to look at the intertemporal hedging demand which is now possible as there is a level of final wealth at the end of each time period. As the CRRA-utility function is concave, the investor prefers a smooth income stream. The intertemporal elasticity of substitution (IES) is directly related to the coefficient of risk aversion in those models and is equal to $\psi = \frac{1}{\gamma}$. As a result, the more risk averse an investor is the less willing he is to accept swings in the level of final wealth over time. The concept of intertemporal hedging is deeply discussed and applied in Campbell & Viceira (2002)[49]. This effect can also be observed in table 5.2. As the level of correlation is lower than the bounds derived above ($\rho = 0.5 < \frac{\sigma_S}{\sigma_P}$), the total weight of risky assets is (slightly) larger in τ_2. But as risk aversion (γ) grows and the intertemporal hedging demand falls, the ratio of risky assets over both time periods becomes more equal. But the size of this effect is very small. This finding is similar to Campbell & Viceira (p.43) who discovered that the intertemporal elasticity of substitution has only minor influences on portfolio choice and is more important when it comes to determine consumption paths over time.

[49] The intertemporal hedging is also especially important in macroeconomic models determining the optimal level of consumption and savings over time, e.g. in the Ramsey-Growth-Model or the Diamond-Overlapping-Generations-Model.

Risk Aversion	Time Period	$t_1 = 3$ Delay	$t_1 = 3$ No Delay	$t_2 = 5$ Delay	$t_2 = 5$ No Delay
$\gamma = 2$	τ_1	22.95%	70.39%	25.18%	70.39%
		45.23%	35.37%	44.77%	35.37%
		31.82%	-5.76%	30.05%	-5.76%
	τ_2	76.49%	70.39%	83.95%	70.39%
		-2.99%	35.37%	-8.16%	35.37%
		26.50%	-5.76%	24.21%	-5.76%
$\gamma = 6$	τ_1	7.65%	23.46%	8.39%	23.46%
		15.08%	11.79%	14.92%	11.79%
		77.27%	64.75%	76.68%	64.75%
	τ_2	25.50%	23.46%	27.98%	23.46%
		-1.00%	11.79%	-2.72%	11.79%
		75.50%	64.75%	74.74%	64.75%
$\gamma = 10$	τ_1	4.59%	14.08%	5.04%	14.08%
		9.05%	7.07%	8.95%	7.07%
		86.36%	78.85%	86.01%	78.85%
	τ_2	15.30%	14.08%	16.79%	14.08%
		-0.60%	7.07%	-1.63%	7.07%
		85.30%	78.85%	84.84%	78.85%
Sum		100%	100%	100%	100%

Table 5.2: Comparison of optimal portfolio weights

6 Conclusion

It becomes very clear that the consideration of Private Equity specific investment characteristics changes the optimal behavior of an investor to a large extent and leads to a suboptimal portfolio as the Private Equity part cannot be reoptimized over time due to its illiquid nature. As the investment flow into Private Equity investments takes place within several investment rounds, the simplifying assumption that committed capital equals invested capital leads to a massive overinvestment in risky assets and can therefore not be neglected. The models derived in this study not only optimize the investment of committed but not yet invested capital; they also determine the optimal weights of assets in a well diversified portfolio. As a result, the models lead to investment rules that improve standard models to a large extent in terms of the expected utility of wealth. It became obvious in this study that the lack of Private Equity specific investment characteristics in portfolio optimization models leads to investment rules that reduce an investor's overall expected utility substantially. The associated opportunity cost from delayed investment into Private Equity funds can reduce the overall annual expected return of Private Equity by several percentage points up to 4% p.a. for realistic parameter choices. Optimal treatment of the committed but not yet invested capital can reduce those cost to a large extent as was shown using the model derived in this study. As Private Equity is mostly part of a well diversified portfolio, we were able to analytically derive an optimization rule for the investment of the gap. An estimation of the expected commitment path into a Private Equity fund can be used to transform the two-period model derived above into a multi-period model. The model can also be extended by incorporating the individual characteristics of an investor's portfolio as the two extensions mentioned within the discussion of the model show. A further step will be the inclusion of state variables driving the economy into the model setting, either fully or partially observable. Very interesting from a practitioner's point of view would be the introduction of various asset sub-classes to the model, like domestic and international shares, small-cap and mid-cap shares, corporate and government bonds, long-term bonds etc. to evaluate their weights within this model context. In case an investor invests only in Private Equity, we have shown that the shortfall probability may be substantial. For that case the model can no more be solved analytically. Using Monte Carlo simulation we have estimated the optimal investment path for those cases. Sensitivities to the model parameters can also be estimated using this framework. An interesting point would be to evaluate trading strategies reducing the shortfall probability like options, futures or other hedging strategies.

Furthermore, the analysis has shown the impact of several terms and conditions of a Private Equity fund on an investor's optimization behavior. Uncertainty around the investment path will drive the portfolio away from the optimal solution derived in this model. As a result, an investor should try to make sure that the fund manager has an incentive to invest according to predefined rules, i.e. stick to a predefined investment period. Implications of the analysis in this study on the optimal structure of terms and conditions would be an interesting question for further research and the evaluation of optimal negotiation strategies.

A Description of Data

As the California Public Employees' Retirement System (CalPERS) is legally obliged to report on each single Private Equity investment, it is possible to infer various results from the data available. The data analyzed here is available from:

http://www.calpers.ca.gov/index.jsp?bc=/investments/assets/equities/aim/private-equity-review/aim-perform-review/home.xml

as of June 30, 2010. CalPERS started its AIM (Alternative Investment Management) Program in June 1990 with an asset allocation target of 14%. Total assets comprise 220 Billion USD. The actual weight of Private Equity commitments is 23.34%. The weighted average age of funds is 4.6 years and more than 50% of the funds was invested in during the last 5 years. The whole investment comprises 632 funds with total commitments of 52.98 Billion USD. CalPERS manages 265 funds with 45.1 Billion USD of committed capital itself. The rest is invested in fund-of-funds with a total number of funds ranging between 9 to 188. The data on these fund-of-funds is available in detail on the website, as well. The majority of funds is invested in buyout funds consistent with the average investment weights in the overall market. A detailed overview is given in figure A.1. The following tables A.1 and A.2 provides an overview on the data. The term *Average IRR* in table A.2 refers to the arithmetic mean of the IRRs of all funds in the respective vintage year. As was explained in section 2.2.2, the IRRs during the investment phase (2006-2010) are not meaningful due to the J-curve effect.

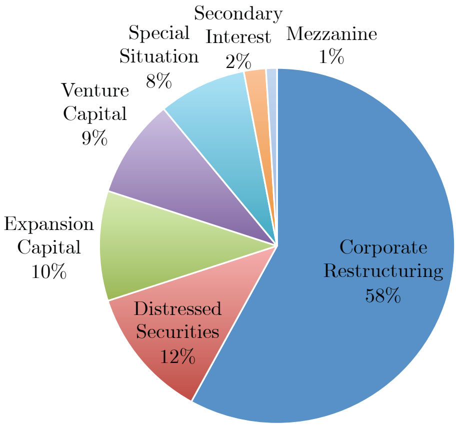


Figure A.1: CalPERS portfolio

Vintage Year	No. of Funds	Commitment (in mio USD)	Avg. Ratio of Investment	S.D. of Inv. Ratio
1990	1	100	100.00%	n.a.
1991	2	113	89.26%	n.a.
1992	1	35	100.00%	n.a.
1993	2	125	100.06%	n.a.
1994	7	313	101.91%	n.a.
1995	6	805	91.89%	n.a.
1996	8	388	98.74%	n.a.
1997	4	401	98.80%	n.a.
1998	11	1'091	100.67%	6.91%
1999	29	788	95.43%	4.52%
2000	61	2'542	93.17%	8.07%
2001	55	3'527	96.91%	5.65%
2002	25	1'052	97.58%	3.41%
2003	22	1'422	89.07%	9.83%
2004	28	1'595	88.73%	8.98%
2005	49	3'632	89.92%	13.40%
2006	59	8'560	77.08%	12.75%
2007	118	13'798	58.66%	21.81%
2008	107	11'559	41.11%	21.94%
2009	23	651	25.52%	26.71%
2010	14	479	15.42%	12.05%
SUM/AVG.	632	52'977	72.97%	1.93%

Table A.1: CalPERS dataset (1)

Vintage Year	Overcommitment Ratio	Funds with Inv.Ratio >90%	Funds with Inv.Ratio >100%
1990	0.00%	100.00%	100.00%
1991	0.50%	50.00%	50.00%
1992	0.00%	100.00%	100.00%
1993	0.06%	100.00%	100.00%
1994	1.98%	100.00%	85.71%
1995	0.00%	66.67%	33.33%
1996	0.37%	100.00%	50.00%
1997	0.48%	100.00%	50.00%
1998	2.83%	90.91%	54.55%
1999	0.14%	89.66%	51.72%
2000	0.26%	81.97%	32.79%
2001	0.59%	83.64%	49.09%
2002	0.40%	96.00%	40.00%
2003	0.11%	50.00%	13.64%
2004	0.12%	60.71%	14.29%
2005	0.22%	42.86%	8.16%
2006	0.00%	11.86%	1.69%
2007	0.00%	14.41%	10.17%
2008	0.00%	4.67%	1.87%
2009	0.00%	8.70%	4.35%
2010	0.00%	0.00%	0.00%
SUM/AVG.	0.36%	41.77%	19.62%

Table A.2: CalPERS dataset (2)

Vintage Year	Average IRR	Avg. IRR Capital Weighted
1990	14.90%	14.90%
1991	26.80%	23.63%
1992	25.50%	25.50%
1993	12.20%	12.04%
1994	15.03%	17.13%
1995	15.70%	11.59%
1996	9.49%	10.21%
1997	36.20%	30.20%
1998	-0.74%	3.46%
1999	5.92%	10.40%
2000	4.34%	11.73%
2001	8.80%	14.63%
2002	9.02%	13.85%
2003	27.10%	22.17%
2004	9.32%	15.29%
2005	6.82%	7.06%
2006	-3.81%	-3.14%
2007	0.86%	0.67%
2008	-6.40%	2.03%
2009	-11.76%	4.36%
2010	-43.27%	5.18%
AVERAGE	7.71%	12.05%

Table A.3: CalPERS dataset (3)

B Institutional Issues

B.1 CalPERS: Alternative Investment Policy

The following statements are an excerpt from the CalPERS' Statement of Investment Policy for Alternative Investment Management. The full publication is available from
http://www.calpers.ca.gov/index.jsp?bc=/investments/policies/inv-asset-classes/aim/ home.xml.

...

Strategic Objective

To maximize risk-adjusted rates of return while enhancing the CalPERS position as a premier alternative investment manager is the strategic objective of the Program.

The Program shall be managed to accomplish the following:

1. Enhance CalPERS long-term total risk-adjusted return;
2. Enhance CalPERS reputation as a premier alternative investment manager and "investor of choice" within the private equity community;
3. Hedge against long-term liabilities; and
4. Provide diversification to the CalPERS overall investment program.

...

Investment Approach

"Top down" strategic assessments shall identify portfolio weightings and identify the most attractive segments of the market for investing. Based on these assessments, the staff shall proactively seek out the most attractive investment opportunities, while maintaining appropriate diversification.

...

Specific Risk Parameters

Valuation: Partnerships and co-investments shall be evaluated to determine if the general partner employs an appropriate valuation discipline. For direct investments, the staff shall review valuations to determine if they are reasonable.

...

B.2 Term Sheet Example

The following terms and conditions are adapted from Connor (2005a, p. 65) and reflect industry standards for funds with a capitalization around 100 mio USD:

- *Term:* 10 years with up to additional two-year extension
- *Investment Period:* Five years
- *Management Fee:* 2% of committed capital paid in advance during the investment period, rate decreases by 0.20% per year until the tenth year, and then ceases
- *Preferred Return:* 8% based on invested capital net of management fees
- *Clawback:* Upon termination of the fund, the General Partners will be required to restore funds to the Limited Partners if the Limited Partners have failed to receive the greater of (a) their total invested capital plus management fees paid and (b) 80% of all profits distributed. Under no circumstances will the clawback amount exceed the amount of total distributions received by the General Partners.

- *Distributions:* Distributions will be made in the following order of priority and amount:
    1. First 100% to the Limited Partners until they have received an amount equal to their invested capital plus management fees paid (return of capital and fees)
    2. Then 100% to the Limited Partners until their net internal rate of return, including management fees paid and the fund's current valuation, has reached the Preferred Return
    3. Then 80% to the General Partners and 20% to the Limited Partners until the General Partners have received 20% of all profits distributed in excess of (1) (catch-up)
    4. Then 80% to the Limited Partners and 20% to the General Partners (carried interest)

Table B.1: Terms and Conditions

C Mathematical Derivations

C.1 Wealth Dynamics

In this Appendix the transformation from 3.27 to 3.29 will be derived.

Expanding each fraction with the corresponding funds in each asset class at T leads to

$$
\begin{aligned}
&\mathrm{d}ln(W_t) = \\
&\frac{C_T}{C_T+S_T+B_T}\frac{dC_t}{C_t}+\frac{S_T}{C_T+S_T+B_T}\frac{dS_t}{S_t}+\frac{B_T}{C_T+S_T+B_T}\frac{dB_t}{B_t} \\
&-\frac{1}{2}\frac{C_T^2}{(C_T+S_T+B_T)^2}(\frac{dC_t}{C_t})^2-\frac{1}{2}\frac{S_T^2}{(C_T+S_T+B_T)^2}(\frac{dS_t}{S_t})^2 \\
&-\frac{1}{2}\frac{B_T^2}{(C_T+S_T+B_T)^2}(\frac{dB_t}{B_t})^2-\frac{C_tS_t}{(C_T+S_T+B_T)^2}\frac{dC_t}{C_t}\frac{dS_t}{S_t}
\end{aligned}
\tag{C.1}
$$

As the square/product of the stochastic differential equations equals the square of the respective stochastic components with $(dZ)^2 = dt$, we get from 3.21 to 3.23

$$
\begin{aligned}
&(\frac{dC_t}{C_t})^2 = \sigma_P^2 dt \\
&(\frac{dS_t}{S_t})^2 = \sigma_S^2 dt \\
&(\frac{dB_t}{B_t})^2 = 0 \\
&\frac{dC_t}{C_t}\frac{dS_t}{S_t} = \sigma_P\sigma_S\rho dt
\end{aligned}
\tag{C.2}
$$

It is obvious that the fractions of the investment amounts in equation C.1 equal the corresponding portfolio weights. Replacing these with the expressions in 3.24 to 3.26 and plugging in the corresponding stochastic differential equations gives the dynamics of log-wealth in equation 3.29:

$$\begin{aligned} \mathrm{d}ln(W_t) = [x\mu_P + s\mu_S + (1-x-s)r - \frac{1}{2}x^2\sigma_P^2 - \frac{1}{2}s^2\sigma_S^2 - xs\sigma_P\sigma_S\rho]dt \\ + x\sigma_P dZ_{P,t} + s\sigma_S dZ_{S,t} \end{aligned} \tag{C.3}$$

C.2 Expected Utility of Optimal Portfolio

In this Appendix the transformation from 3.31 to 3.33 will be derived. The expectation of utility of final wealth is given by

$$\begin{aligned} E[u(W_t)] = \frac{1}{1-\gamma} e^{(1-\gamma)(x\mu_P + s\mu_S + (1-x-s)r - \frac{1}{2}x^2\sigma_P^2 - \frac{1}{2}s^2\sigma_S^2 - xs\sigma_P\sigma_S\rho)T} \\ \underbrace{E[e^{(1-\gamma)(x\sigma_P Z_{P,T} + s\sigma_S Z_{S,T})}]}_{\text{Stochastic Components}} \end{aligned} \tag{C.4}$$

Calculating the stochastic part yields

$$\begin{aligned} E[e^{(1-\gamma)(x\sigma_P Z_{P,T} + s\sigma_S Z_{S,T})}] = e^{\frac{1}{2}Var((1-\gamma)(x\sigma_P Z_{P,T} + s\sigma_S Z_{S,T}))} \\ = e^{\frac{1}{2}(1-\gamma)^2(x^2\sigma_P^2 Var(Z_{P,T}) + s^2\sigma_S^2 Var(Z_{S,T}) + 2xs\sigma_P\sigma_S Cov(Z_{P,T}, Z_{S,T}))} \end{aligned} \tag{C.5}$$

As Z follows $Z_T \sim N(0, \sqrt{T})$ and private and public equity have a correlation of ρ, the expected value equals

$$E[u(W_t)] = \frac{1}{1-\gamma}\exp[(1-\gamma)(x\mu_P + s\mu_S + (1-x-s)r - \frac{1}{2}x^2\sigma_P^2 - \frac{1}{2}s^2\sigma_S^2 - xs\sigma_P\sigma_S\rho)T + \frac{1}{2}(1-\gamma)^2(x^2\sigma_P^2 T + s^2\sigma_S^2 T + 2xs\sigma_P\sigma_S\rho T)] \tag{C.6}$$

As the maximization of expected final wealth is equivalent to maximizing the exponent dividing by the constant term $(1-\gamma)$, we end up with

$$\begin{aligned} &\max_{x,s}\ E[u(W_T)] \\ \Longrightarrow\ &\max_{x,s}[(x\mu_P + s\mu_S + (1-x-s)r - \frac{1}{2}x^2\sigma_P^2 - \frac{1}{2}s^2\sigma_S^2 - xs\sigma_P\sigma_S\rho)T \\ &+ \frac{1}{2}(1-\gamma)(x^2\sigma_P^2 T + s^2\sigma_S^2 T + 2xs\sigma_P\sigma_S\rho T)] \\ &= (x\mu_P + s\mu_S + (1-x-s)r - \frac{1}{2}\gamma x^2\sigma_P^2 - \frac{1}{2}\gamma s^2\sigma_S^2 - \gamma xs\sigma_P\sigma_S\rho)T \end{aligned} \tag{C.7}$$

C.3 Opportunity Cost without Shortfall

In the following Appendix, the expression for the expected utility of final wealth is solved. Expected utility from final wealth (equation 4.15) for the effective investment path is

$$\begin{aligned} E[u(W_T)] = &\frac{1}{1-\gamma}\exp[(1-\gamma)(\mu_P - \frac{1}{2}\sigma_P^2)T + (1-\gamma)(\delta\mu_P - \mu_P - \frac{1}{2}\delta^2\sigma_P^2 \\ &+ \frac{1}{2}\sigma_P^2 + (1-\delta)\mu_S - (1-\delta)^2\sigma_S^2 - \delta(1-\delta)\sigma_P\sigma_S\rho)t_1] \\ &\underbrace{E[e^{(1-\gamma)(\delta\sigma_P Z_{P,t_1} + (1-\delta)\sigma_S Z_{S,t_1} + \sigma_P(Z_{P,T} - Z_{P,t_1}))}]}_{\text{Stochastic Components}} \end{aligned} \tag{C.8}$$

As all deterministic components can be taken out of the expectation, we are left with calculating the expectation of the stochastic part. The Brownian motion has independent increments and Z follows $Z \sim N(0, \sqrt{t})$. This leads to

$$e^{\frac{1}{2}\text{Var}((1-\gamma)(\sigma_P(Z_{P,T}-Z_{P,t_1})+\delta\sigma_P Z_{P,t_1}+(1-\delta)\sigma_S Z_{S,t_1}))} = e^{\frac{1}{2}(1-\gamma)^2[\sigma_P^2(T-t_1)+\delta^2\sigma_P^2 t_1+(1-\delta)^2\sigma_S^2 t_1+2\delta(1-\delta)\sigma_P\sigma_S\rho t_1]} \tag{C.9}$$

The resulting expression of expected utility of effective final wealth equal to equation 4.16 therefore is

$$E[u(W_T)] = \frac{1}{1-\gamma}\exp[(1-\gamma)(\mu_P - \frac{1}{2}\sigma_P^2)T - (1-\gamma)(1-\delta)(\mu_P - \mu_S - \frac{1}{2}(1+\delta)\sigma_P^2 + \frac{1}{2}(1-\delta)\sigma_S^2 - \delta\sigma_P\sigma_S\rho)t_1 + \frac{1}{2}(1-\gamma)^2(\sigma_P^2(T-t_1) + \delta^2\sigma_P^2 t_1 + (1-\delta)^2\sigma_S^2 t_1 + 2\delta(1-\delta)\sigma_P\sigma_S\rho t_1)] \tag{C.10}$$

In order to solve for the opportunity cost p, we have to solve equation 4.8

$$E[u(W_T)] = E[u(e^{-pT}\bar{W_T})] \tag{C.11}$$

Equating equations C.10 and 4.14, multiplying by $(1-\gamma)$, taking logs and then dividing by $(1-\gamma)$ leads to

$$(\mu_P - p - \frac{1}{2}\sigma_P^2)T + \frac{1}{2}(1-\gamma)\sigma_P^2 T = (\mu_P - \frac{1}{2}\sigma_P^2)T - (1-\delta)(\mu_P - \mu_S - \frac{1}{2}(1+\delta)\sigma_P^2 + \frac{1}{2}(1-\delta)\sigma_S^2 - \delta\sigma_P\sigma_S\rho)t_1 + \frac{1}{2}(1-\gamma)(\sigma_P^2(T-t_1) + \delta^2\sigma_P^2 t_1 + (1-\delta)^2\sigma_S^2 t_1 + 2\delta(1-\delta)\sigma_P\sigma_S\rho t_1) \tag{C.12}$$

Now solving for p is straightforward and gives us the result in equation 4.17:

$$p = \frac{(1-\delta)t_1}{T}[\mu_P - \mu_S - \frac{1}{2}\gamma(1+\delta)\sigma_P^2 + \frac{1}{2}\gamma(1-\delta)\sigma_S^2 + \gamma\delta\sigma_P\sigma_S\rho] \quad \text{(C.13)}$$

C.4 Opportunity Cost with Certain Default

In this Appendix, opportunity cost in case of a certain default are derived. From equation 4.38, with the changing dynamics in τ_2 and including the cost of a default on commitment, α, $E[u(W_T)]$ becomes:

$$\begin{aligned} E[u(W_T)] &= \frac{1}{1-\gamma}\exp[(1-\gamma)((1-\pi_M)r + \pi_M\mu_S - \frac{1}{2}\pi_M^2\sigma_S^2)(T-t_1) \\ &+ (1-\gamma)(\bar{\delta}\mu_P - \frac{1}{2}\bar{\delta}^2\sigma_P^2 + (1-\bar{\delta})\mu_S - \frac{1}{2}(1-\bar{\delta})^2\sigma_S^2 - \bar{\delta}(1-\bar{\delta})\sigma_P\sigma_S\rho)t_1 \\ &+ (1-\gamma)ln(1-(1-\alpha)\delta)] \\ &\underbrace{E[e^{(1-\gamma)(\pi_M\sigma_S(Z_{S,T}-Z_{S,t_1})+\bar{\delta}\sigma_P Z_{P,t_1}+(1-\bar{\delta})\sigma_S Z_{S,t_1})}]}_{\text{Stochastic Components}} \end{aligned} \quad \text{(C.14)}$$

Calculating the expectation of the stochastic part is similar to the derivation in Appendix C.3 and leads to

$$e^{\frac{1}{2}(1-\gamma)^2(\pi_M^2\sigma_S^2(T-t_1)+\bar{\delta}^2\sigma_P^2 t_1+(1-\bar{\delta})^2\sigma_S^2 t_1+2\bar{\delta}(1-\bar{\delta})\sigma_P\sigma_S\rho t_1)} \quad \text{(C.15)}$$

The resulting expected utility of final wealth in case of a default on commitment therefore is

$$E[u(W_T)] = \frac{1}{1-\gamma}\exp[(1-\gamma)((1-\pi_M)r + \pi_M\mu_S - \frac{1}{2}\pi_M^2\sigma_S^2)(T-t_1)$$
$$+ (1-\gamma)(\bar{\delta}\mu_P - \frac{1}{2}\bar{\delta}^2\sigma_P^2 + (1-\bar{\delta})\mu_S - \frac{1}{2}(1-\bar{\delta})^2\sigma_S^2 - \bar{\delta}(1-\bar{\delta})\sigma_P\sigma_S\rho)t_1$$
$$+ ln(1-(1-\alpha)\delta) + \frac{1}{2}(1-\gamma)^2(\pi_M^2\sigma_S^2(T-t_1) + \bar{\delta}^2\sigma_P^2 t_1 + (1-\bar{\delta})^2\sigma_S^2 t_1$$
$$+ 2\bar{\delta}(1-\bar{\delta})\sigma_P\sigma_S\rho t_1)] \tag{C.16}$$

Equating equations C.16 and 4.14 and multiplying by $(1-\gamma)$, taking logs and then dividing by $(1-\gamma)$ leads to

$$(\mu_P - p - \frac{1}{2}\gamma\sigma_P^2)T =$$
$$((1-\pi_M)r + \pi_M\mu_S - \frac{1}{2}\pi_M^2\sigma_S^2)(T-t_1) + (\bar{\delta}\mu_P - \frac{1}{2}\bar{\delta}^2\sigma_P^2 + (1-\bar{\delta})\mu_S$$
$$- \frac{1}{2}(1-\bar{\delta})^2\sigma_S^2 - \bar{\delta}(1-\bar{\delta})\sigma_P\sigma_S\rho)t_1 + ln(1-(1-\alpha)\delta)$$
$$+ \frac{1}{2}(1-\gamma)(\pi_M^2\sigma_S^2(T-t_1) + \bar{\delta}^2\sigma_P^2 t_1 + (1-\bar{\delta})^2\sigma_S^2 t_1 + 2\bar{\delta}(1-\bar{\delta})\sigma_P\sigma_S\rho t_1) \tag{C.17}$$

Solving for p gives the opportunity cost when a default on commitment occurs

$$p = \mu_P - \pi_M\mu_S - (1-\pi_M)r - \frac{1}{2}\gamma(\sigma_P^2 - \pi_M^2\sigma_S^2) + \frac{ln(1-(1-\alpha)\delta)}{T}$$
$$+ (\pi_M\mu_S + (1-\pi_M)r - \frac{1}{2}\gamma\pi_M^2\sigma_S^2 - \bar{\delta}\mu_P + \frac{1}{2}\bar{\delta}^2\gamma\sigma_P^2 - (1-\bar{\delta})\mu_S$$
$$+ \frac{1}{2}(1-\bar{\delta})^2\gamma\sigma_S^2 + \bar{\delta}(1-\bar{\delta})\gamma\sigma_P\sigma_S\rho)\frac{t_1}{T} \tag{C.18}$$

To introduce π, the following changes have to be made in the above equation: $\mu_S \rightarrow (1-\pi)r + \pi\mu_S$ and $\sigma_S \rightarrow \pi\sigma_S$. Minimizing with respect to π leads to the optimal investment weights of the investment gap when a default on commitment occurs.

C.5 Wealth Dynamics in Mixed Portfolio

In this Appendix the transformation from equations 5.13 to 5.15 will be derived.

At first, we cancel out $\bar{W}_0$ and expand each fraction with the corresponding funds in each asset class at t, $\frac{\bar{C}_t}{\bar{C}_t}$, $\frac{S_t}{S_t}$ and $\frac{B_t}{B_t}$, respectively. This leads to

$$\begin{aligned}
\mathrm{d}ln(W_t) = &\frac{\bar{C}_t}{\bar{C}_t + S_t + B_t}\frac{d\bar{C}_t}{\bar{C}_t} + \frac{B_t}{\bar{C}_t + S_t + B_t}\frac{dB_t}{B_t} + \frac{S_t}{\bar{C}_t + S_t + B_t}\frac{dS_t}{S_t} \\
&- \frac{1}{2}\frac{\bar{C}_t^2}{(\bar{C}_t + S_t + B_t)^2}(\frac{d\bar{C}_t}{\bar{C}_t})^2 - \frac{1}{2}\frac{B_t^2}{(\bar{C}_t + S_t + B_t)^2}(\frac{dB_t}{B_t})^2 \\
&- \frac{1}{2}\frac{S_t^2}{(\bar{C}_t + S_t + B_t)^2}(\frac{dS_t}{S_t})^2 - \frac{1}{2}\frac{\bar{C}_t}{(\bar{C}_t + S_t + B_t)}\frac{S_t}{(\bar{C}_t + S_t + B_t)}2\frac{d\bar{C}_t}{\bar{C}_t}\frac{dS_t}{S_t}
\end{aligned} \tag{C.19}$$

As the square/product of the stochastic differential equations equals the square of the respective stochastic components with $(dZ)^2 = dt$, we get from equations 5.10, 4.2 and 4.3

$$\begin{aligned}
(\frac{d\bar{C}_t}{\bar{C}_t})^2 &= \pi^2\sigma_S^2 dt \\
(\frac{dS_t}{S_t})^2 &= \sigma_S^2 dt \\
(\frac{dB_t}{B_t})^2 &= 0 \\
\frac{d\bar{C}_t}{\bar{C}_t}\frac{dS_t}{S_t} &= \pi\sigma_S^2 dt
\end{aligned} \tag{C.20}$$

It is obvious that the fractions of the investment amounts in equation C.19 equal the corresponding portfolio weights. Replacing those with the expressions in 5.7 to 5.9 and plugging in the stochastic differential equations gives 5.15

$$\begin{aligned} \mathrm{d}ln(W_t) = {} & [(1-\pi)\bar{x}r + \pi\bar{x}\mu_S - \frac{1}{2}\bar{x}^2\pi^2\sigma_S^2 + \bar{s}\mu_S - \frac{1}{2}\bar{s}^2\sigma_S^2 + \bar{b}r - \bar{x}\bar{s}\pi\sigma_S^2]dt \\ & + (\pi\bar{x} + \bar{s})\sigma_S dZ_{S,t} \end{aligned} \tag{C.21}$$

C.6 Expected Utility in Mixed Portfolio

To calculate $E[u(W_T)]$ from equation 5.44 we have to plug in final wealth into the utility function (CRRA) from equation 3.14 and calculate the expectation. The expectation of the deterministic parts is constant, therefore only the expectation of the stochastic parts is calculated in detail here:

$$\begin{aligned} & E[e^{(1-\gamma)(x\sigma_P(Z_{P,T}-Z_{P,t_1})+s_2\sigma_S(Z_{S,T}-Z_{S,t_1})+\delta x\sigma_P Z_{P,t_1}+s_1\sigma_S Z_{S,t_1})}] = \\ & \exp[\frac{1}{2}(1-\gamma)^2(x^2\sigma_P^2(T-t_1) + s_2^2\sigma_S^2(T-t_1) + \delta^2x^2\sigma_P^2t_1 + s_1^2\sigma_S^2t_1 \\ & + 2xs_2\sigma_P\sigma_S\rho(T-t_1) + 2\delta xs_1\sigma_P\sigma_S\rho t_1)] \end{aligned} \tag{C.22}$$

Rearranging leads to

$$\begin{aligned} & \exp[\frac{1}{2}(1-\gamma)^2((x^2\sigma_P^2 + s_2^2\sigma_S^2 + 2xs_2\sigma_P\sigma_S\rho)T - (1-\delta^2)x^2\sigma_P^2t_1 \\ & - (s_2^2 - s_1^2)\sigma_S^2t_1 - 2(s_2 - \delta s_1)x\sigma_P\sigma_S\rho t_1)] \end{aligned} \tag{C.23}$$

C.7 Solving FOCs for Mixed Portfolio

The optimal weights can be calculated explicitly from the first order conditions (FOC) in equation 5.49 using Cramer's rule. Following this rule, the optimal weights are

$$
\begin{aligned}
x^* &= \frac{1}{\gamma}\frac{\det V_1}{\det V} \\
s_2^* &= \frac{1}{\gamma}\frac{\det V_2}{\det V} \\
s_1^* &= \frac{1}{\gamma}\frac{\det V_3}{\det V}
\end{aligned}
\tag{C.24}
$$

where the index in the denominator indicates the column in the covariance matrix that has to be replaced by the vector on the right hand side of equation 5.49. This leads to

$$
\begin{aligned}
&\det V = \\
&\sigma_P^2\sigma_S^4(T-(1-\delta^2)t_1)(T-t_1)t_1 - \delta^2\sigma_P^2\sigma_S^4\rho^2(T-t_1)t_1^2 \\
&-\sigma_P^2\sigma_S^4\rho^2(T-t_1)^2t_1
\end{aligned}
\tag{C.25}
$$

$$
\begin{aligned}
&\det V_1 = \\
&(\mu_P-r)\sigma_S^4(T-(1-\delta)t_1)(T-t_1)t_1 - (\mu_S-r)\delta\sigma_S^3\sigma_P\rho(T-t_1)t_1^2 \\
&-(\mu_S-r)\sigma_S^3\sigma_P\rho(T-t_1)^2t_1
\end{aligned}
\tag{C.26}
$$

$$
\begin{aligned}
&\det V_2 = \\
&(\mu_S-r)\sigma_P^2\sigma_S^2(T-(1-\delta^2)t_1)(T-t_1)t_1 + (\mu_S-r)\delta\sigma_P^2\sigma_S^2\rho^2(T-t_1)t_1^2 \\
&-(\mu_P-r)\sigma_P\sigma_S^3\rho(T-(1-\delta)t_1)(T-t_1)t_1 - (\mu_S-r)\delta^2\sigma_P^2\sigma_S^2\rho^2(T-t_1)t_1^2
\end{aligned}
\tag{C.27}
$$

$$
\begin{aligned}
&\det V_3 = \\
&(\mu_S-r)\sigma_P^2\sigma_S^2(T-(1-\delta^2)t_1)(T-t_1)t_1 + (\mu_S-r)\delta\sigma_P^2\sigma_S^2\rho^2(T-t_1)^2t_1 \\
&-(\mu_S-r)\sigma_P^2\sigma_S^2\rho^2(T-t_1)^2t_1 - (\mu_P-r)\delta\sigma_P\sigma_S^3\rho(T-(1-\delta)t_1)(T-t_1)t_1
\end{aligned}
\tag{C.28}
$$

Plugging in these expressions into Cramer's rule, dividing each fraction by $\sigma_P^2\sigma_S^2(T-t_1)t_1$ and rearranging leads to the results in equations 5.51 to 5.53:

$$x^* = \frac{(\mu_P - r - \frac{\mu_S - r}{\sigma_S}\sigma_P\rho)(T - (1-\delta)t_1)}{\gamma\sigma_P^2(1-\rho^2)(T-(1-\delta^2)t_1)} \tag{C.29}$$

$$s_1^* = \frac{(\mu_S - r)\frac{\sigma_P^2}{\sigma_S^2}(T - (1-\delta^2)t_1 - \rho^2(1-\delta)(T - t_1))}{\gamma\sigma_P^2(1-\rho^2)(T-(1-\delta^2)t_1)} - \frac{(\mu_P - r)\frac{\sigma_P}{\sigma_S}\delta\rho(T-(1-\delta)t_1)}{\gamma\sigma_P^2(1-\rho^2)(T-(1-\delta^2)t_1)} \tag{C.30}$$

$$s_2^* = \frac{(\mu_S - r)\frac{\sigma_P^2}{\sigma_S^2}(T - (1-\delta^2)t_1 + \delta(1-\delta)\rho^2 t_1) - (\mu_P - r)\frac{\sigma_P}{\sigma_S}\rho(T-(1-\delta)t_1)}{\gamma\sigma_P^2(1-\rho^2)(T-(1-\delta^2)t_1)} \tag{C.31}$$

References

Amenc, N. & Martellini, L. (2002, Fall). Portfolio Optimization and Hedge Fund Style Allocation Decisions. *Journal of Alternative Investments*.

Bader, H. (1996). *Private Equity als Anlagekategorie.* Bern: Haupt.

Baierl, G., Chen, P. & Kaplan, P. (2002, Winter). Venture Capital and its Role in Strategic Asset Allocation. *Journal of Portfolio Management*, 83-89.

Bailey, M., Muth, R. & Nourse, H. (1963, December). A Regression Method for Real Estate Price Index Construction. *Journal of the American Statistical Society*, *58*(304), 993-942.

Baks, K., Metrick, A. & Wachter, J. (2001, February). Should Investors Avoid All Actively Managed Funds? A Study in Bayesian Performance Evaluation. *Journal of Finance*, *56*(1), 45-85.

Baumann, R. & Müller, H. (2006). Shortfall Minimizing Portfolios. *SAV Bulletin*, *2*, 125-142.

Bellman, R. (1957). *Dynamic Programming.* Princeton: University Press.

Bilo, S. (2002). *Alternative Asset Class: Publicly Traded Private Equity - Performance, Liquidity, Diversification Potential and Pricing Characteristics.* Dissertation, University of St. Gallen.

Bjørk, T. (1998). *Arbitrage Theory in Continuous Time.* New York: Oxford University Press.

Black, F. & Litterman, R. (1992). Global Portfolio Optimization. *Financial Analysts Journal*, *48*(5), 28-43.

Breiman, L. (1961). Optimal Gambling Systems for Favorable Games. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, University of California Press*, *1*, 65-78.

Brennan, M. (1998). The Role of Learning in Dynamic Portfolio Decisions. *European Finance Review*, *1*, 295-306.

Brennan, M. & Xia, Y. (2001, Winter). Assessing Asset Pricing Anomalies. *Review of Financial Studies*, *14*(4), 905-942.

Campbell, J., Chan, Y. & Viceira, L. (2003). A Multivariate Model of Strategic Asset Allocation. *Journal of Financial Economics*, *67*, 41-80.

Campbell, J. & Viceira, L. (2002). *Strategic Asset Allocation - Portfolio Choice for Long-Term Investors* (2nd ed.). New York: Oxford University Press.

Cardie, J., Cattanach, K. & Kelley, M. (2000). How large should your Commitment to Private Equity really be. *Journal of Wealth Management*, *4*, 39-45.

Cochrane, J. (2005). The Risk and Return of Venture Capital. *Journal of Financial Economics*, *75*, 3-52.

Connor, A. (2005a, Summer). The Economic Value of Terms and Conditions: What is Worth Fighting For? *Journal of Private Equity*, 64-73.

Connor, A. (2005b, March). Persistence in Venture Capital Returns. *Private Equity International*, 65-67.

Connor, A. (2005c, July). Properties of Capital Draws in Private Equity Funds. *Research Forum ACG*. Available from http://www.alignmentcapital.com/research.htm

Connor, A., Nevins, D. & McIntire, G. (2004, Spring). A Portfolio Management Approach to Determining Private Equity Commitments. *Journal of Alternative Investments*, 32-46.

Cox, J. & Huang, C. (1989). Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process. *Journal of Economic Theory*, *49*, 33-83.

Cox, J., Ingersoll, J. & Ross, S. (1985, March). An Intertemporal General Equilibrium Model of Asset Prices. *Econometrica*, *53*(2), 363-384.

Cvitanic, J. & Karatzas, I. (1992). Convex Duality in Constrained Portfolio Optimization. *Annals of Applied Probability*, *2*(4), 767-818.

Cvitanic, J., Lazrak, A., Martellini, L. & Zapatero, F. (2003a). Optimal Allocation to Hedge Funds: An Empirical Analysis. *Quantitative Finance*, *3*, 28-39.

Cvitanic, J., Lazrak, A., Martellini, L. & Zapatero, F. (2003b). Revisiting Treynor and Black (1973): An Intertemporal Model of Active Portfolio Management. *Working Paper, University of Southern California.*

Dean, T. & Seth, A. (2002, Winter). Illiquid Alternative Asset Fund Modelling. *Journal of Portfolio Management.*

Deng, X.-T., Wang, S.-Y. & Xia, Y.-S. (2000). Criteria, Models and Strategies in Portfolio Selection. *AMO - Advanced Modeling and Optimization*, *2*(2), 79-103.

Denzler, M., Müller, H. & Scherer, D. (2001). A Practical Application of Continuous Time Finance: Calculation of Benchmark Portfolios. *Mitteilungen der Schweizerischen Aktuarvereinigung*, *2*, 139-164.

Dondi, G., Geering, H., Herzog, F. & Schumann, L. (2004). Continuous-Time Multivariate Strategic Asset Allocation. Available from http://www.uni-tuebingen.de/dgf/program/Cpaper114.pdf

Døskeland, T. (2007). Strategic asset allocation for a country: the Norwegian case. *Financial Markets and Portfolio Management*, *21*, 167-201.

Driessen, J., Lin, T.-C. & Phalippou, L. (2008). A New Method to Estimate Risk and Return of Non-Traded Assets from Cash Flows: The Case of Private Equity Funds. *NBER Working Paper Series*, *14144*.

Fama, E. & French, K. (1993). Common Risk Factors in the Returns on Stocks and Bonds. *Journal of Financial Economics*, *33*, 3-56.

Frei, A. & Studer, M. (2003). Quantitative Private Equity Risk Management. *Partners Goup Research Series*. Available from http://www.partnersgroup.net/index.php?nav=204,273,260,261

Frei, A. & Studer, M. (2004). Payback Time. *Partners Goup Research Series*. Available from http://www.partnersgroup.net/index.php?nav=204,273,260,261

Gilligan, J. & Wright, M. (2010). *Private Equity Demystified* (2nd ed.). ICAEW.

Gompers, P. & Lerner, J. (1996, October). The Use of Covenants: An Empirical Analysis of Venture Partner Agreements. *Journal of Law* *and Economics*, *39*(2), 463-498.

Gottschalg, O. & Phalippou, L. (2006). The Performance of Private Equity Funds. *Working Paper, University of Amsterdam*. Available from http://papers.ssrn.com/sol3/papers.cfm?abstract_id=473221

Hall, R. & Woodward, S. (2003). Benchmarking the Returns to Venture. *NBER Working Paper Series*, *10202*.

Harrison, M. & Pliska, S. (1981). Martingales and Stochastic Integrals in the Theory of Continuous Trading. *Stochastic Processes and Their Applications*, *11*, 215-260.

Hwang, M., Quigley, J. & Woodward, S. (2005). An Index For Venture Capital, 1987-2003. *Contributions to Economic Analysis & Policy*, *4*(1), Article 13.

Janecek, K. (2004). What is a realistic Aversion to Risk for real-world individual Investors? *Working Paper, Carnegie Mellon University*, 1-20.

Jensen, M. (1968, May). The Performance of Mutual Funds in the Period 1945-1964. *Journal of Finance*, *23*(2), 389-416.

Kamien, M. & Schwartz, N. (1992). *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management* (2nd ed.). New York: North Holland.

Kaplan, S. & Schoar, A. (2005). Private Equity Performance: Returns, Persistence and Capital Flows. *Journal of Finance*, *60*, 1791-1823.

Karatzas, I. (1989). Optimization Problem in the Theory of Contiuous Trading. *SIAM Journal of Control and Optimization*, *27*, 1221-1259.

Karatzas, I. & Shreve, S. (1991). *Brownian Motion and Stochastic Calculus* (2nd ed.). New York: Springer.

Keel, S. (2006). *Optimal Portfolio Construction and Active Portfolio Management Including Alternative Investments.* Dissertation, ETHZ.

Kerins, F., Smith, J. & Smith, R. (2004). Opportunity Cost of Capital for Venture Capital Investors and Entrepreneurs. *Journal of Financial and Quantitative Analysis*, *39*(2), 385-405.

Kim, T. & Omberg, E. (1996, Spring). Dynamic Nonmyopic Portfolio Behaviour. *Review of Financial Studies*, *9*(1), 141-161.

King, D. (2007). Portfolio optimization and diversification. *Journal of Asset Management*, *8*(5), 296-307.

Korn, R. (1997). *Optimal Portfolios - Stochastic Models for Optimal Investments and Risk Management in Continuous Time*. Singapore: World Scientific.

Korn, R. & Schäl, M. (1999). On growth optimal and value preserving portfolio. *Mathematical Methods of Operations Research*, *50*(2), 189-218.

Korteweg, A. & Sorensen, M. (2007). Estimating Risk and Return of Infrequently-Traded Assets: A Bayesian Selection Model of Venture Capital. *Working Paper*.

Lenoir, G. & Tuchschmid, N. (2001). Investment Time Horizon and Asset Allocation Models. *Financial Markets and Portfolio Management*, *15*(1), 76-93.

Lintner, J. (1965). The Valuation of Risky Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. *Review of Economics and Statistics*, *47*, 13-37.

Ljungqvist, A. & Richardson, M. (2003). The Cash Flow, Return and Risk Characteristics of Private Equity. *NBER Working Paper Series*, *9454*.

Long, A. (1999, Summer). Inferring Periodic Variability of Private Market Returns as Measured by σ from the Range of Value (Wealth) Outcomes over Time. *Journal of Private Equity*, 63-69.

Long, A. (2006, November). A Method for Estimating the Change in Terminal Value Required to Increase IRR. *Research Forum ACG*. Available from http://www.alignmentcapital.com/research.htm

Long, A. (2008, January). The Common Mathematical Foundation of ACG's ICM and AICM and the K&S PME. *Research Forum ACG*. Available from http://www.alignmentcapital.com/research.htm

Long, A. & Nickels, C. (1996). A Private Investment Benchmark. *AIMR - Conference on Venture Capital Investing*.

Long, J. (1990). The numeraire portfolio. *Journal of Financial Economics*, *26*, 29-69.

Longstaff, F. (2001, Summer). Optimal Portfolio Choice and the Valuation of Illiquid Securities. *Review of Financial Studies*, *14*(2), 407-431.

Lucas, R. (1978, November). Asset Prices in an Exchange Economy. *Econometrica*, *46*(6), 1429-1445.

Mankiw, G. & Zeldes, S. (1991). The Consumption of Stockholders and Nonstockholders. *Journal of Financial Economics*, *29*, 97-112.

Markowitz, H. (1952). Portfolio Selection. *Journal of Finance*, *7*(1), 77-91.

Mehra, R. & Prescott, E. (1985). The Equity Premium - A Puzzle. *Journal of Monetary Economics*, *15*, 145-1611.

Merton, R. (1971). Optimum Consumption and Portfolio Rules in a Continuous-Time Model. *Journal of Economic Theory*, *3*, 373-413.

Merton, R. (1973). An Intertemporal Capital Asset Pricing Model. *Econometrica*, *41*(5), 867-887.

Metrick, A. & Yasuda, A. (2008). The Economics of Private Equity Funds. *Mimeo Wharton School, University of Pennsylvania*. Available from http://papers.ssrn.com/sol3/papers.cfm ?abstract_id=996334&rec=1&srcabs=980243

Moskowitz, T. & Vissing-Jørgensen, A. (2002, September). The Returns to Entrepreneurial Investment: A Private Equity Premium Puzzle. *American Economic Review*, *92*(4), 745-778.

Mossin, J. (1966). Equilibrium in a Capital Asset Market. *Econometrica*, *34*, 768-783.

Munk, C. (2008). Dynamic Asset Allocation. *Working Paper, Copenhagen Business School*.

Munk, C. & Sørensen, C. (2004). Optimal Consumption and Investment Strategies with Stochastic Interest Rates. *Journal of Banking and Finance*, *28*, 1987-2013.

Peng, L. (2001). Building a venture capital index. *Working Paper, Yale University*(51).

Phalippou, L. (2007). Investing in Private Equity Funds: A Survey. *Research Foundation of CFA Institute*. Available from http://papers.ssrn.com/sol3/papers.cfm?abstract_id=

980243&rec=1&srcrabs=965917

Platen, E. (2005). On the Role of the Growth Optimal Portfolio in Finance. *Australian Economic Papers*, *44*(4), 365-388.

Prequin. (2010a). 2010 Private Equity Fundraising. *Prequin Research Report*. Available from http://www.preqin.com/listResearch.aspx

Prequin. (2010b). Prequin Investor Outlook: Private Equity. Available from http://www.preqin.com/listResearch.aspx

Ross, S. (1976). The arbitrage theory of capital asset pricing. *Journal of Economic Theory*, *13*(3), 341-360.

Samuelson, P. (1969). Lifetime Portfolio Selection by Stochastic Dynamic Programming. *Review of Economics and Statistics*, *51*(3), 239-246.

Sharpe, W. (1963). A Simplified Model for Portfolio Analysis. *Management Science*, *9*, 277-293.

Sharpe, W. (1964). Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. *Journal of Finance*, *19*, 425-442.

Sharpe, W. (1966, January). Mutual Fund Performance. *Journal of Business*, *39*, 119-138.

Shreve, S. (2004). *Stochastic Calculus and Finance*. New York: Springer.

Sørensen, C. (1999). Dynamic Asset Allocation and Fixed Income Management. *Journal of Financial and Quantitative Analysis*, *34*(4), 513-531.

Swensen, D. (2000). *Pioneering Portfolio Management*. New York: The Free Press.

Tobin, J. (1965a). Money and Economic Growth. *Econometrica*, *33*, 671-684.

Tobin, J. (1965b). *The Theory of Portfolio Selection* (F. Hahn & F. Brenchling, Eds.). London: Macmillan.

Treynor, J. & Black, F. (1973, January). How to Use Security Analysis to Improve Portfolio Selection. *Journal of Business*, *46*(1), 66-86.

Xia, Y. (2001, February). Learning about Predictability: The Effects of Parameter Uncertainty on Dynamic Asset Allocation. *Journal of Finance*, *56*(1), 205-246.

Yale. (2005). *The Yale Endowment*. Yale University.

Yale. (2009). *The Yale Endowment*. Yale University.

Zwart, G. de, Frieser, B. & Dijk, D. van. (2007, December). A Recommitment Strategy for Long Term Private Equity Fund Investors. *ERIM Report Series Research in Management*, *ERS-2007-097-F&A*.

Curriculum Vitae

Mattias Thomas Fürstenberger, M.A. HSG

Born 2nd October 1981 in Lahr/Schwarzwald, Germany

Education

2007-2011	Doctoral Studies in Finance University of St. Gallen (HSG)
2005-2007	Master Studies in Economics University of St. Gallen (HSG) China Europe International Business School (CEIBS), Shanghai
2002-2005	Bachelor Studies in Economics University of St. Gallen (HSG) Singapore Management University (SMU), Singapore

Work Experience

2009-2011	Bonafide Wealth Management AG Co-Fund Manager
2008-2011	University of Applied Sciences St. Gallen (FHS) Lecturer in Mathematics
2007-2011	WaKa Holding AG Analyst
2005-2011	Institute of Mathematics & Statistics, University of St. Gallen Assistant
2007	UBS AG Fixed Income Recommendations